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Bipartite algebraic graphs without quadrilaterals

Boris Bukh [a], Zilin Jiang [b,*]

[a] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
[b] Department of Mathematics, the Technion – Israel Institute of Technology, Technion City, Haifa 3200003, Israel

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ABSTRACT

Let $\mathbb{P}^s$ be the s-dimensional complex projective space, and let X, Y be two non-empty open subsets of $\mathbb{P}^s$ in the Zariski topology. A hypersurface H in $\mathbb{P}^s \times \mathbb{P}^s$ induces a bipartite graph G as follows: the partite sets of G are X and Y, and the edge set is defined by $\overline{u} \sim \overline{v}$ if and only if $(\overline{u}, \overline{v}) \in H$. Motivated by the Turán problem for bipartite graphs, we say that $H \cap (X \times Y)$ is (s, t)-grid-free provided that G contains no complete bipartite subgraph that has s vertices in X and t vertices in Y. We conjecture that every (s, t)-grid-free hypersurface is equivalent, in a suitable sense, to a hypersurface whose degree in $\overline{y}$ is bounded by a constant $d = d(s, t)$, and we discuss possible notions of the equivalence.

We establish the result that if $H \cap (X \times \mathbb{P}^2)$ is $(2, 2)$-grid-free, then there exists $F \in \mathbb{C}[\overline{x}, \overline{y}]$ of degree ≤ 2 in $\overline{y}$ such that $H \cap (X \times \mathbb{P}^2) = \{F = 0\} \cap (X \times \mathbb{P}^2)$. Finally, we transfer the result to algebraically closed fields of large characteristic.

1. Introduction

The Turán number $\operatorname{ex}(n, F)$ is the maximum number of edges in an F-free graph[1] on n vertices. The first systematic study of $\operatorname{ex}(n, F)$ was initiated by Turán [26], who solved the case when $F = K_t$ is a complete graph on t vertices. Turán's theorem states that, on a given vertex set, the K_t-free graph with the most edges is the complete and balanced $(t-1)$-partite graph, in that the part sizes are as equal as possible.

For general graphs F, we still do not know how to compute the Turán number exactly, but if we are satisfied with an approximate answer, the theory becomes quite simple: Erdős and Stone [8] showed that if the chromatic number $\chi(F) = t$, then $\operatorname{ex}(n, F) = \operatorname{ex}(n, K_t) + o(n^2) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2)$. When F is not bipartite, this gives an asymptotic result for the Turán number. On the other hand, for all but few bipartite graphs F, the order of $\operatorname{ex}(n, F)$ is not known. Most of the research on this problem focused on two classes of graphs: complete bipartite graphs and cycles of even length. A comprehensive survey is given by Füredi and Simonovits [13].

Suppose G is a $K_{s,t}$-free graph with $s \leq t$. The Kövari–Sós–Turán theorem [17] implies an upper bound $\operatorname{ex}(n, K_{s,t}) \leq \frac{1}{2}\sqrt[s]{t-1} \cdot n^{2-1/s} + o(n^{2-1/s})$, which was improved by Füredi [11] to

$$\operatorname{ex}(n, K_{s,t}) \leq \frac{1}{2}\sqrt[s]{t-s+1} \cdot n^{2-1/s} + o(n^{2-1/s}).$$

Despite the lack of progress on the Turán problem for complete bipartite graphs, there are certain complete bipartite graphs for which the problem has been solved asymptotically, or even exactly. The constructions that match the upper bounds in these cases are all similar to one another. Each of the constructions is a bipartite graph G based on an algebraic hypersurface[2]

* Corresponding author.
E-mail address: jiangzilin@technion.ac.il (Z. Jiang).

[1] We say a graph is F-free if it does not have a subgraph isomorphic to F.

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H. Both partite sets of G are $\mathbb{F}_p^s$ and the edge set is defined by: $\bar{u} \sim \bar{v}$ if and only if $(\bar{u}, \bar{v}) \in H$. In short, $G = \big(\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p)\big)$, where $H(\mathbb{F}_p)$ denotes the $\mathbb{F}_p$-points of H. Note that G has $n := 2p^s$ vertices.

In the previous works of Erdős, Rényi and Sós [7], Brown [5], Füredi [10], Kollár, Rónyai and Szabó [16] and Alon, Rónyai and Szabó [2], various hypersurfaces were used to define $K_{s,t}$-free graphs. Their equations were

$$x_1y_1 + x_2y_2 = 1, \quad \text{for } K_{2,2}; \tag{1a}$$

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1, \quad \text{for } K_{3,3}; \tag{1b}$$

$$(N_s \circ \pi_s)(x_1 + y_1, x_2 + y_2, \ldots, x_s + y_s) = 1, \quad \text{for } K_{s,t} \text{ with } t \geq s! + 1; \tag{1c}$$

$$(N_{s-1} \circ \pi_{s-1})(x_2 + y_2, x_3 + y_3, \ldots, x_s + y_s) = x_1y_1, \quad \text{for } K_{s,t} \text{ with } t \geq (s-1)! + 1, \tag{1d}$$

where $\pi_s : \mathbb{F}_p^s \to \mathbb{F}_{p^s}$ is an $\mathbb{F}_p$-linear isomorphism and $N_s(\alpha)$ is the field norm, $N_s(\alpha) := \alpha^{(p^s-1)/(p-1)}$.

Clearly, the coefficients in (1a) and (1b) are integers and even independent of p. With some work, one can show that both (1c) and (1d) are polynomial equations of degree $\leq s$ with coefficients in $\mathbb{F}_p$. Therefore each equation in (1) can be written as $F(\bar{x}, \bar{y}) := F(x_1, \ldots, x_s, y_1, \ldots, y_s) = 0$ for some $F(\bar{x}, \bar{y}) \in \mathbb{F}_p[\bar{x}, \bar{y}]$ of bounded degree. The previous works directly count the number of $\mathbb{F}_p$ solutions to $F(\bar{x}, \bar{y}) = 0$ and yield $|H(\mathbb{F}_p)| = \Theta(p^{2s-1}) = \Theta(n^{2-1/s})$, for each prime[3] p.

Definition 1. Given two sets P_1 and P_2, a set $V \subset P_1 \times P_2$ is said to contain an (s, t)-grid if there exist $S \subset P_1$, $T \subset P_2$ such that $s = |S|$, $t = |T|$ and $S \times T \subset V$. Otherwise, we say that V is (s, t)-grid-free.

Observe that every $F(\bar{x}, \bar{y})$ derived from (1) is symmetric in x_i and y_i for all i. We know that $(\bar{u}, \bar{v}) \in H$ if and only if $(\bar{v}, \bar{u}) \in H$ for all $\bar{u}, \bar{v} \in \mathbb{F}_p^s$. The resulting bipartite graph $G = \big(\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p)\big)$ would be an extremal $K_{s,t}$-free graph if $H(\mathbb{F}_p)$ had been (s, t)-grid-free.

So which graphs are $K_{s,t}$-free with a maximum number of edges? The question was considered by Zoltán Füredi in his unpublished manuscript [9] asserting that every $K_{2,2}$-free graph with q vertices (for $q \geq q_0$) and $\frac{1}{2}q(q+1)^2$ edges is obtained from a projective plane via a polarity with $q+1$ absolute elements. This loosely amounts to saying that all extremal $K_{2,2}$-free graphs are defined by generalization of (1a).

However, classification of all extremal $K_{s,t}$-free graphs seems out of reach. We restrict our attention to algebraically constructed graphs. Given a field $\mathbb{F}$ and a hypersurface H defined over $\mathbb{F}$, it is natural to ask when $H(\mathbb{F})$ is (s, t)-grid-free. Because the general case is difficult, we work with algebraically closed fields $\mathbb{K}$ in this paper. Denote by $\mathbb{P}^s(\mathbb{K})$ the s-dimensional projective space over $\mathbb{K}$. We are interested in hypersurface H in $\mathbb{P}^s(\mathbb{K}) \times \mathbb{P}^s(\mathbb{K})$.

Since standard machinery from model theory, to be discussed in Section 5, allows us to transfer certain results over $\mathbb{C}$ (the field of complex numbers) to algebraically closed fields of large characteristic, our focus will be on the $\mathbb{K} = \mathbb{C}$ case. We use $\mathbb{P}^s$ for the s-dimensional complex projective space and $\mathbb{A}^s := \mathbb{P}^s \setminus \{x_0 = 0\}$ for the s-dimensional complex affine space.

Note that even if H contains (s, t)-grids, one may remove a few points from the projective space to destroy all (s, t)-grids in H. For example, the homogenization of (1b) is

$$(x_1y_0 - x_0y_1)^2 + (x_2y_0 - x_0y_2)^2 + (x_3y_0 - x_0y_3)^2 = x_0^2y_0^2.$$

The equation defines hypersurface H in $\mathbb{P}^3 \times \mathbb{P}^3$. Let $V := \{x_0 = x_1^2 + x_2^2 + x_3^2 = 0\}$ be a variety in $\mathbb{P}^3$. Since $V \times \mathbb{P}^3 \subset H$, H contains a lot of $(3, 3)$-grids. However, $H \cap (\mathbb{A}^3 \times \mathbb{A}^3)$ is $(3, 3)$-grid-free.

Definition 2. A set $V \subset \mathbb{P}^s \times \mathbb{P}^s$ is almost-(s, t)-grid-free if there are two nonempty Zariski-open sets $X, Y \subset \mathbb{P}^s$ such that $V \cap (X \times Y)$ is (s, t)-grid-free.

Suppose the defining equation of H, say $F(\bar{x}, \bar{y})$, is of low degree in $\bar{y}$. Heuristically, for generic[4] distinct $\bar{u}_1, \ldots, \bar{u}_s \in \mathbb{P}^s$, by Bézout's theorem, one would expect $\{F(\bar{u}_1, \bar{y}) = \cdots = F(\bar{u}_s, \bar{y}) = 0\}$ to have few points. So we conjecture the following.

Informal conjecture. *Every almost-(s, t)-grid-free hypersurface is equivalent, in a suitable sense, to a hypersurface whose degree in $\bar{y}$ is bounded by some constant $d := d(s, t)$.*

The right equivalence notion depends on X and Y in Definition 2. We shall discuss possible notions of equivalence in Section 2, and make three specific conjectures. Results in support of these conjectures can be found in Section 3 and Section 4.

Before we make our conjectures precise, we note that an analogous situation occurs for C_{2t}-free graphs. The upper bound $\mathrm{ex}(n, C_{2t}) = O(n^{1+1/t})$ first established by Bondy–Simonovits [4] has been matched only for $t = 2, 3, 5$. The $t = 2$ case was already mentioned above because $C_4 = K_{2,2}$. The constructions for $t = 3, 5$ are also algebraic (see [3,12] for $t = 3$ and [3,27] for $t = 5$). Also, a conjecture in a similar spirit about algebraic graphs of girth eight was made by Dmytrenko, Lazebnik and Williford [6]. It was recently resolved by Hou, Lappano and Lazebnik [14].

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[2] An algebraic hypersurface in a space of dimension n is an algebraic subvariety of dimension $n - 1$. The terminology from algebraic geometry used throughout the article is standard, and can be found in [23].

[3] We need $p \equiv 3 \pmod 4$ for (1b) to get the correct number of $\mathbb{F}_p$ points on H. If $p \equiv 1 \pmod 4$, then the right hand side of (1b) should be replaced by a quadratic non-residue in $\mathbb{F}_p$.

[4] Henceforth, a statement is true for a generic point $\bar{u} \in \mathbb{P}^s$ means that there exists a nonempty Zariski-open set $U \subset \mathbb{P}^s$ such that the statement is true for every $\bar{u} \in U$.