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On the metric dimension of incidence graphs

Robert F. Bailey

School of Science & Environment (Mathematics), Grenfell Campus, Memorial University of Newfoundland, Corner Brook, NL A2H 6P9, Canada

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ABSTRACT

A *resolving set* for a graph Γ is a collection of vertices S, chosen so that for each vertex v, the list of distances from v to the members of S uniquely specifies v. The *metric dimension* $\mu(\Gamma)$ is the smallest size of a resolving set for Γ. We consider the metric dimension of two families of incidence graphs: incidence graphs of symmetric designs, and incidence graphs of symmetric transversal designs (i.e. symmetric nets). These graphs are the bipartite distance-regular graphs of diameter 3, and the bipartite, antipodal distance-regular graphs of diameter 4, respectively. In each case, we use the probabilistic method in the manner used by Babai to obtain bounds on the metric dimension of strongly regular graphs, and are able to show that $\mu(\Gamma) = O(\sqrt{n}\log n)$ (where n is the number of vertices).

1. Introduction

We consider finite, connected graphs with no loops or multiple edges. Let Γ denote a graph with vertex set V and edge set E. A *resolving set* for Γ is a subset $S \subseteq V$ with the property that, for any $u \in V$, the list of distances from u to each of the elements of S uniquely identifies u; equivalently, for two distinct vertices $u, w \in V$, there exists $x \in S$ for which $\mathrm{d}(u, x) \neq \mathrm{d}(w, x)$. The *metric dimension* of Γ is the smallest size of a resolving set for Γ, and we denote this by $\mu(\Gamma)$. These notions were introduced to graph theory in the 1970s by Slater [28] and, independently, Harary and Melter [24]; in more general metric spaces, the concept can be found in the literature much earlier (see [13]). For further details, the reader is referred to the survey [8].

When studying metric dimension, distance-regular graphs are a natural class of graphs to consider. A graph Γ with diameter d is *distance-regular* if, for all i with $0 \leq i \leq d$ and any vertices u, w with $\mathrm{d}(u, w) = i$, the number of neighbours of w at distances $i-1$, i and $i+1$ from u depend only on the distance i, and not on the choices of u and w. These numbers are denoted by c_i, a_i and b_i, respectively, and are known as the *parameters* of Γ. It is easy to see that c_0, b_d are undefined, $a_0 = 0$, $c_1 = 1$ and $c_i + a_i + b_i = k$ (where k is the valency of Γ). We put the parameters into an array, called the *intersection array* of Γ,

$$\begin{Bmatrix} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_{d-1} & * \end{Bmatrix}.$$

Since the 2011 survey article by Cameron and the present author [8], which first proposed its systematic study, a number of papers have been written on the subject of the metric dimension of distance-regular graphs (and on the related problem of class dimension of association schemes), by the present author and others: see [5–7,9–11,17–23,25], for instance; earlier

E-mail address: rbailey@grenfell.mun.ca.

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results may be found in [3,4,15,16,27]. For background on distance-regular graphs in general, see the book of Brouwer, Cohen and Neumaier [14] or the survey by van Dam, Koolen and Tanaka [29].

A distance-regular graph Γ with diameter d is *primitive* if, for $1 \leq i \leq d$, the distance-i graphs of Γ are all connected; otherwise, we say it is *imprimitive*. Imprimitive distance-regular graphs arise in one of two ways, provided that the valency is at least 3: they may be bipartite (whereby the distance-2 graph has two connected components, called the *halved graphs* of Γ), or *antipodal* (where the distance-d graph is a disjoint union of cliques). We note that both possibilities may occur in the same graph. Imprimitive distance-regular graphs may be reduced to primitive ones by the operations of *halving* (for bipartite graphs) or *folding* (for antipodal graphs); see [14, § 4.2A] for details. If an imprimitive graph Γ has diameter $d \geq 3$, its halved or folded graphs have diameter $\lfloor d/2 \rfloor$.

The metric dimension of imprimitive distance-regular graphs was studied in detail in [6], where it was shown that it can be bounded in terms of the metric dimension of the halved or folded graphs (see [6, § 2.1]). However, when the halved or folded graphs are either complete or complete multipartite, the results are unsatisfactory; this is especially true from the asymptotic perspective, as we obtain the trivial upper bound of $O(n)$ (where n is the number of vertices). In this paper, we consider bipartite distance-regular graphs of diameter 3, and distance-regular graphs of diameter 4 which are both bipartite and antipodal. The former class is precisely equivalent to the incidence graphs of *symmetric designs*, which are well-understood objects (see [26], for instance); the latter class is equivalent to the incidence graphs of *symmetric transversal designs*, or equivalently *symmetric nets*, about which the literature is more sporadic.

1.1. Split resolving sets and semi-resolving sets

In [6], the present author introduced the following special type of resolving set for bipartite graphs.

Definition 1.1. Let Γ be a bipartite graph, whose vertex set has bipartition $X \cup Y$. A *split resolving set* for Γ is a subset of vertices $S = S_X \cup S_Y$, where $S_X \subseteq X$ and $S_Y \subseteq Y$, chosen so that any two vertices in X are resolved by a vertex in S_Y, and any two vertices in Y are resolved by a vertex in S_X. We call S_X a *semi-resolving set* for Y and S_Y a semi-resolving set for X. We denote the smallest size of a split resolving set by $\mu^*(\Gamma)$.

We note that a split resolving set is itself a resolving set: any vertex of Γ will resolve a pair of vertices (x, y) where $x \in X$ and $y \in Y$, given that the parities of the distances to x and to y will be different, so we only need consider resolving pairs of vertices in the same bipartite half. Consequently, we have $\mu(\Gamma) \leq \mu^*(\Gamma)$. We also note that complete bipartite graphs do not have split resolving sets.

If we regard a bipartite graph Γ as an incidence graph, semi-resolving sets are of independent interest due to connections with other objects associated with incidence structures, such as blocking sets in finite geometries; see [6,10,25] for more details on this.

2. Symmetric designs

A *symmetric design* (or *square* 2-*design*) with parameters (v, k, λ) is a pair $\mathcal{D} = (X, \mathcal{B})$, where X is a set of v *points*, and $\mathcal{B}$ is a family of k-subsets of X, called *blocks*, such that any pair of distinct points are contained in exactly λ blocks, and that any pair of distinct blocks intersect in exactly λ points. It follows that $|\mathcal{B}| = v$. A symmetric design with $\lambda = 1$ is a *projective plane*, while a symmetric design with $\lambda = 2$ is known as a *biplane*. The *incidence graph* $\Gamma_{\mathcal{D}}$ of a symmetric design $\mathcal{D}$ is the bipartite graph with vertex set $X \cup \mathcal{B}$, with the point $x \in X$ adjacent to the block $B \in \mathcal{B}$ if and only if $x \in B$. It is straightforward to show that the incidence graph of a symmetric design is a bipartite distance-regular graph with diameter 3 and intersection array

$$\begin{Bmatrix} * & 1 & \lambda & k \\ 0 & 0 & 0 & 0 \\ k & k-1 & k-\lambda & * \end{Bmatrix}.$$

The converse is also true (see [14, § 1.6]): any bipartite distance-regular graph of diameter 3 gives rise to a symmetric design. The *dual* of a symmetric design is the design obtained from the incidence graph by reversing the roles of points and blocks; both $\mathcal{D}$ and its dual have the same parameters.

The *order* of a symmetric design is defined to be $q = k - \lambda$; the following result is well-known (see [26, Proposition 2.4.12], for instance) and gives restrictions on v in terms of the order.

Proposition 2.1. *For any* (v, k, λ) *symmetric design of order* $q = k - \lambda \geq 2$, *we have*

$$4q - 1 \leq v \leq q^2 + q + 1.$$

The two extremes are achieved by Hadamard designs (where $v = 4q - 1$) and projective planes (where $v = q^2 + q + 1$).

The incidence graphs of symmetric designs are precisely the bipartite distance-regular graphs of diameter 3; the metric dimension of these graphs is considered in [6]. However, the general results of [6] for bipartite distance-regular graphs are not very effective in the diameter 3 case, as the halved graphs are complete graphs, so an alternative approach was required. First, in the case where $k = v - 1$ (or, equivalently, where the order is $q = k - \lambda = 1$), the incidence graph is $K_{v,v} - I$,