Discrete Mathematics 341 (2018) 1620–1628

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Dyck paths with a first return decomposition constrained by height

Jean-Luc Baril *, Sergey Kirgizov, Armen Petrossian

LE2I UMR-CNRS 6306, Université de Bourgogne, B.P. 47 870, 21078 DIJON-Cedex, France

ARTICLE INFO

Article history:
Received 6 June 2017
Received in revised form 16 October 2017
Accepted 2 March 2018
Available online 20 March 2018

Keywords:
Enumeration
Dyck and Motzkin paths
First return decomposition
Statistics
Height
Peak

ABSTRACT

We study the enumeration of Dyck paths having a first return decomposition with special properties based on a height constraint. We exhibit new restricted sets of Dyck paths counted by the Motzkin numbers, and we give a constructive bijection between these objects and Motzkin paths. As a byproduct, we provide a generating function for the number of Motzkin paths of height k with a flat (resp. with no flats) at the maximal height.

1. Introduction and notations

A *Dyck path* of semilength $n \geq 0$ is a lattice path starting at $(0, 0)$, ending at $(2n, 0)$, and never going below the x-axis, consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$. Let $\mathcal{D}_n$, $n \geq 0$, be the set of all Dyck paths of semilength n, and let $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$. The cardinality of $\mathcal{D}_n$ is given by the nth Catalan number, which is the general term $\frac{1}{n+1}\binom{2n}{n}$ of the sequence A000108 in the On-line Encyclopedia of Integer Sequences of N.J.A. Sloane [17]. A large number of various classes of combinatorial objects are enumerated by the Catalan numbers such as planar trees, Young tableaux, stack sortable permutations, Dyck paths, and so on. A list of over 60 types of such combinatorial classes has been compiled by Stanley [18]. In combinatorics, many papers deal with Dyck paths. Most of them consist in enumerating Dyck paths according to several parameters, such as length, number of peaks or valleys, number of double rises, number of returns to the x-axis (see for instance [2,8–16,19]). Other studies investigate restricted classes of Dyck paths avoiding some patterns or having a specific structure. For instance, Barcucci et al. [1] consider non-decreasing Dyck paths which are those having a non-decreasing sequence of heights of valleys (see also [6,7]), and it is well known [5] that Dyck paths avoiding the triple rise UUU are enumerated by the Motzkin numbers (see A001006 in [17]).

Any non-empty Dyck path $P \in \mathcal{D}$ has a unique first return decomposition [8] of the form $P = U\alpha D\beta$ where α and β are two Dyck paths in $\mathcal{D}$. See Fig. 1 for an illustration of this decomposition.

Based on this decomposition, we construct a new collection of subsets of $\mathcal{D}$ as follows. Given a function $s : \mathcal{D} \to \mathbb{N}$, called *statistic*, and a comparison operator $\diamond$ on $\mathbb{N}$ (for instance $\geq$ or $>$), the set $\mathcal{D}^{s,\diamond}$ is the union of the empty Dyck path with all Dyck paths P having a first return decomposition $P = U\alpha D\beta$ satisfying the conditions:

$$\begin{cases}\alpha, \beta \in \mathcal{D}^{s,\diamond}, \\ s(U\alpha D) \diamond s(\beta).\end{cases} \tag{1}$$

* Corresponding author.
E-mail addresses: barjl@u-bourgogne.fr (J.-L. Baril), sergey.kirgizov@u-bourgogne.fr (S. Kirgizov), armen.petrossian@u-bourgogne.fr (A. Petrossian).

https://doi.org/10.1016/j.disc.2018.03.002

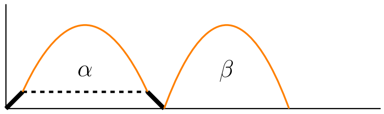

Fig. 1. First return decomposition $U\alpha D\beta$ of a Dyck path $P \in \mathcal{D}$.

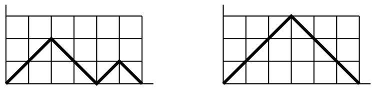

Fig. 2. The two Dyck paths in $\mathcal{D}_3^{h,>}$.

Table 1
Cardinalities of $\mathcal{D}_n^{h,\diamond}$ according to the $\diamond$-constraint.

$\diamond$-constraint	Sequence	OEIS	a_n, $1 \le n \le 9$
$h(U\alpha D) > h(\beta)$		A045761	1, 1, 2, 3, 6, 12, 24, 50, 107
$h(U\alpha D) \ge h(\beta)$	Motzkin	A001006	1, 2, 4, 9, 21, 51, 127, 323, 835

For $n \ge 0$, we denote by $\mathcal{D}_n^{s,\diamond}$ the set of Dyck paths of semilength n in $\mathcal{D}^{s,\diamond}$. Thus, we have $\mathcal{D}^{s,\diamond} = \bigcup_{n\ge 0}\mathcal{D}_n^{s,\diamond}$.

For instance, if the operator $\diamond$ is $=$ and s is a constant statistic (*i.e.*, $s(P) = 0$ for any $P \in \mathcal{D}$), then we obviously have $\mathcal{D}_n^{s,\diamond} = \mathcal{D}_n$ for $n \ge 0$.

If s is the number of returns (*i.e.*, $s(P)$ is the number of down steps D that return the path P to the x-axis) and $s(U\alpha D) \diamond s(\beta)$ is $s(U\alpha D) \ge s(\beta)$, then it is straightforward to see that $\mathcal{D}_n^{s,\ge}$ consists of Dyck paths built over the grammar $S \to \epsilon \mid USD \mid USDUSD$. So, the generating function $S(x)$ for the cardinalities of $\mathcal{D}_n^{s,\ge}$, $n \ge 0$, satisfies the functional equation $S(x) = 1 + xS(x) + x^2S(x)^2$. The solution of this equation is the well-known generating function for the Motzkin numbers (A001006 in [17]).

In this paper, we focus on the sets $\mathcal{D}^{h,\diamond}$ where the statistic h is the maximal height of a Dyck path, *i.e.*, $h(P)$ is the maximal ordinate reached by the path P.

In Section 2, we deal with the case in which operator $\diamond$ is a strict inequality $>$. We prove that the cardinalities of the sets $\mathcal{D}_n^{h,>}$, $n \ge 0$, are given by the sequence A045761 in [17]. This sequence corresponds to the coefficients of the series $\lim_{k\to\infty} P_k(x)$ where $P_k(x)$ is a polynomial recursively defined by $P_0(x) = x$, $P_1(x) = x^2$, $P_k(x) = P_{k-1}(x) + P_{k-2}(x)$ if k is even, and $P_k(x) = P_{k-1}(x) \cdot P_{k-2}(x)$ if k is odd.

In Section 3, we focus on the set $\mathcal{D}^{h,\ge}$ where $h(U\alpha D) \ge h(\beta)$ (the operator $\diamond$ is $\ge$). Using generating functions, we prove that the cardinalities of the $\mathcal{D}_n^{h,\ge}$, $n \ge 0$, are given by the Motzkin numbers (A001006 in [17]). Moreover, we give a constructive one-to-one correspondence ϕ between Dyck paths in $\mathcal{D}_n^{h,\ge}$ and Motzkin paths of length n. Also, we show how ϕ transforms peaks UD into peaks UD and flats F in Motzkin paths. Finally, we deduce generating function for the total number of peaks in $\mathcal{D}_n^{h,\ge}$.

Table 1 presents the two main enumerative results of $\mathcal{D}_n^{h,\diamond}$ obtained in Sections 2 and 3.

2. Enumeration of $\mathcal{D}_n^{h,>}$

In this section, we enumerate the set $\mathcal{D}_n^{h,>}$ of Dyck paths of semilength $n \ge 0$ with a first return decomposition satisfying $h(U\alpha D) > h(\beta)$ where h is the maximal height of a Dyck path. For instance, we have $\mathcal{D}_1^{h,>} = \{UD\}$, $\mathcal{D}_2^{h,>} = \{UUDD\}$, and $\mathcal{D}_3^{h,>} = \{UUDDUD, UUUDDD\}$ (see Fig. 2).

Let $A_k(x) = \sum_{n\ge 0} a_{n,k}x^n$ (resp. $B_k(x) = \sum_{n\ge 0} b_{n,k}x^n$) be the generating function where the coefficient $a_{n,k}$ (resp. $b_{n,k}$) is the number of Dyck paths in $\mathcal{D}_n^{h,>}$ with a maximal height equal to k (resp. of at most k). So, we have $B_k(x) = \sum_{i=0}^{k} A_i(x)$ and the generating function $B(x)$ for the set $\mathcal{D}^{h,>}$ is given by $B(x) = \lim_{k\to+\infty} B_k(x)$.

Any Dyck path of height k in $\mathcal{D}^{h,>}$ is either empty, or of the form $U\alpha D\beta$ where α is a Dyck path in $\mathcal{D}^{h,>}$ of height $k-1$ and $\beta \in \mathcal{D}^{h,>}$ such that $h(\beta) \le k-1$. So, we deduce easily the following functional equations:

$$\begin{cases} A_0(x) & = B_0(x) = 1, \\ A_k(x) & = xA_{k-1}(x) \cdot B_{k-1}(x) \text{ for } k \ge 1. \end{cases} \tag{2}$$

Theorem 1. *We have for* $k \ge 0$,

$$B_k(x) = \frac{P_{2k}(x)}{x}$$