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Maximum matchings in regular graphs

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ABSTRACT

It was conjectured by Mkrtchyan, Petrosyan and Vardanyan that every graph G with $\Delta(G)-\delta(G)\leq 1$ has a maximum matching M such that any two M-unsaturated vertices do not share a neighbor. The results obtained in Mkrtchyan et al. (2010), Petrosyan (2014) and Picouleau (2010) leave the conjecture unknown only for k-regular graphs with $4\leq k\leq 6$. All counterexamples for k-regular graphs ($k\geq 7$) given in Petrosyan (2014) have multiple edges. In this paper, we confirm the conjecture for all k-regular simple graphs and also k-regular multigraphs with $k\leq 4$.

1. Introduction

Graphs considered in this paper may have multi-edges, but no loops. A graph without multi-edges is called a *simple* graph. A *matching* M of a graph G is a set of independent edges. A vertex is M-*saturated* if it is incident with an edge of M, and M-*unsaturated* otherwise. A matching M is said to be *maximum* if for any other matching M', $|M|\geq|M'|$. A matching M is *perfect* if it covers all vertices of G. If G has a perfect matching, the every maximum matching is a perfect matching. The maximum and minimum degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Mkrtchyan, Petrosyan and Vardanyan [4,5] made the following conjecture.

Conjecture 1.1 (*Mkrtchyan et al. [4,5]*)**.** *Let G be a graph with $\Delta(G)-\delta(G)\leq 1$. Then G contains a maximum matching M such that any two M-unsaturated vertices do not share a neighbor.*

This conjecture is verified for subcubic graphs (i.e. $\Delta(G)=3$) with multi-edges by Mkrtchyan, Petrosyan and Vardanyan [5]. Later, Picouleau [7] find a counterexample to the conjecture, which is a bipartite simple graph with $\delta(G)=4$ and $\Delta(G)=5$. Petrosyan [6] constructs counterexamples to the conjecture for all k-regular graphs with $k\geq 7$ and for graphs G with $\Delta(G)-\delta(G)=1$ and $\Delta(G)\geq 4$. Note that, most of counterexamples of Conjecture 1.1 for graphs G with $\Delta(G)-\delta(G)=1$ are simple, but all k-regular graphs with $k\geq 7$ given by Petrosyan [6] have multi-edges. As affirmative answer to Conjecture 1.1 is known only for graphs with $\Delta(G)\leq 3$, Mkrtchyan et al. [5] asked whether the conjecture holds for any k-regular graphs with $k\geq 4$.

In this note, we consider the conjecture for both k-regular simple graphs and k-regular graphs with multi-edges. First we show that Conjecture 1.1 does hold for all k-regular simple graphs.

Theorem 1.2. *Let G be a k-regular simple graph. Then G has a maximum matching M such that any two M-unsaturated vertices do not share a neighbor.*

Further, we show that Conjecture 1.1 holds for k-regular graphs with multi-edges for $k\leq 4$.

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Theorem 1.3. *Let G be a k-regular graph with $k \leq 4$. Then G has a maximum matching M such that any two M-unsaturated vertices do not share a neighbor.*

Our results together with examples given by Petrosyan [6] leave Conjecture 1.1 unknown for 5 and 6-regular graphs with multi-edges.

2. Preliminaries

Let G be a graph and v be a vertex of G. The *neighborhood* of v is set of all vertices adjacent to v, denoted by $N(v)$. The degree of v, denoted by $d_G(v)$ (or $d(v)$ if there is no confusion), is the number of edges incident to v. For $X \subseteq V(G)$, let $\delta(X) := \min\{d(v)|v \in X\}$ and $\Delta(X) := \max\{d(v)|v \in X\}$. The neighborhood of X is defined as $N(X) := \{y|y$ is a neighbor of a vertex $x \in X\}$. For two subsets X_1 and X_2 of $V(G)$, use $[X_1, X_2]$ to denote the all edges with one endvertex in X_1 and another endvertex in X_2. For two subgraphs G_1 and G_2 of G, the symmetric difference of $G_1 \oplus G_2$ is defined as a subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $(E(G_1) \cup E(G_2)) \setminus (E(G_1) \cap E(G_2))$.

A matching of a graph G is a *near-perfect matching* if it covers all vertices except one. If a graph G has a near perfect matching, then G has odd number of vertices. A graph is *factor-critical* if, for any vertex v, the subgraph $G \setminus \{v\}$ has a perfect matching. Every maximum matching of a factor-critical graph is a near-perfect matching.

Let D be the set of all vertices of a graph G which are not covered by at least one maximum matching, and A, the set of all vertices in $V(G) - D$ adjacent to at least one vertex in D. Denote $C = V(G) - A - D$. The graph induced by all vertices in D (resp. A and C) is denoted by $G[D]$ (resp. $G[A]$ and $G[C]$). The following theorem characterizes the structures of maximum matchings of graphs, which is due to Gallai [2] and Edmonds [1].

Theorem 2.1 (*Gallai–Edmonds Structure Theorem, Theorem 3.2.1 in [3]*)**.** *Let G be a graph, and A, D and C are defined as above. Then:*

(1) the components of the subgraph induced by D are factor-critical;

(2) the subgraph induced by C has a perfect matching;

(3) if M is a maximum matching of G, it contains a near-perfect matching of each component of $G[D]$, a perfect matching of $G[C]$ and matches all vertices of A with vertices in distinct components of $G[D]$.

Let M be a maximum matching of a graph G. By Gallai–Edmonds Structure Theorem, M does not contain edges from $G[A]$ and all M-unsaturated vertices of G belong to D. Contract every component of $G[D]$ to a vertex and let B be the set of all these vertices. Let $G(A, B)$ be the bipartite graph with bipartition (A, B) and all edges of G in $[A, D]$. Because every component of $G[D]$ is factor-critical, a maximum matching of $G(A, B)$ is corresponding to a maximum matching of G, and vice versa. In fact, the proofs of our results mainly focus on maximum matchings of the bipartite graph $G(A, B)$.

Before processing to prove our main results, we need some results for maximum matchings of bipartite graphs $G(A, B)$.

Theorem 2.2 (*Hall's Theorem, Theorem 1.13 in [3]*)**.** *Let $G(A, B)$ be a bipartite graph. If $|N(S)| \geq |S|$ for any $S \subseteq A$, then G has a matching M covering all vertices of A.*

The following technical lemma is needed in the proofs of our main results.

Lemma 2.3. *Let $G(A, B)$ be a bipartite graph such that every maximum matching of $G(A, B)$ covers all vertices of A. Let $W \subseteq B$ such that $\delta(W) \geq \Delta(A) \neq 0$. Then $G(A, B)$ has a maximum matching M covering all vertices of W.*

Proof. Let M be a maximum matching of $G(A, B)$ such that the number of vertices of W covered by M is maximum. If M covers all vertices of W, the lemma follows. So assume that there exists an M-unsaturated vertex $x \in W$.

For any $U \subseteq W$, we have $\delta(U) \geq \delta(W)$ and $N(U) \subset A$. Further,

$$\delta(W)|U| \leq \delta(U)|U| \leq |[U, N(U)]| \leq \sum_{v \in N(U)} d(v) \leq \Delta(A)|N(U)|.$$

It follows that $|N(U)| \geq |U|$ because $\delta(W) \geq \Delta(A) \neq 0$. By applying Hall's Theorem on the subgraph induced by W and $N(W)$, it follows that G has a matching M' covering all vertices of W.

Let $M \oplus M'$ be the symmetric difference of M and M'. Every component of $M \oplus M'$ is either a path or a cycle. Since x is not covered by M but is covered by M', it follows that x is an end-vertex of some path-component P of $M \oplus M'$. Let y be another end-vertex of P. Note that every vertex of A is covered by an edge of M and every vertex of W is covered by an edge of M'. So $y \in B \backslash W$.

Then let $M'' = M \oplus P$. Then M'' is a maximum matching of G which covers x and all vertices covered by M except y. Note that $y \in B \backslash W$ and $x \in W$. Hence M'' covers more vertices of W than M, a contradiction to the maximality of the number of vertices of W covered by M. This completes the proof. □