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Patterns in the generalized Fibonacci word, applied to games

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ABSTRACT

We analyze a natural generalization, $\mathcal{W}$, of the infinite Fibonacci word over the alphabet $\Sigma = \{a, b\}$. We provide tools to represent explicitly the set $\{s \in \mathbb{Z}_{\geq 0} : \mathcal{W}(s) = b,\ \mathcal{W}(s+x) = a\}$ for any fixed positive integer x. We show how this representation can be used to analyze the preservation of P-positions of any game whose P-positions are a pair of complementary Beatty sequences, in particular a certain generalization of Wythoff Nim (Holladay, 1968; Fraenkel, 1982).

1. Introduction

1.1. The generalized Fibonacci word

Consider the alphabet $\Sigma = \{a, b\}$. Starting with the letter a, and iteratively replacing a with ab and b with a ($a \to ab \to aba \to abaab \to \cdots$), one obtains the well known infinite Fibonacci word, $\mathcal{F} = \mathcal{F}(0)\mathcal{F}(1)\mathcal{F}(2)\cdots = abaababaabaab\cdots$. See, for example, [19, ch. 1].

It is known that the positions of the a's are given by $\{\lfloor \alpha n \rfloor - 1 : n \geq 1\}$ and the positions of the b's are given by the complementary sequence $\{\lfloor \beta n \rfloor - 1 : n \geq 1\}$, where here α is the golden ratio and $\beta = \alpha + 1$.

The infinite Fibonacci word can be generalized to any irrational $\alpha > 1$: The two sequences $\{\lfloor \alpha n \rfloor - 1 : n \geq 1\}$ and $\{\lfloor \beta n \rfloor - 1 : n \geq 1\}$, where $1/\alpha + 1/\beta = 1$, are a pair of complementary Beatty sequences (see [2,8,10]). Thus, there is a unique infinite word, $\mathcal{W} = \mathcal{W}[\alpha]$, for which the positions of the a's are given by the first sequence and the positions of the b's are given by the second. Throughout the paper, α and β will always be positive and irrational, and will satisfy the equality $1/\alpha + 1/\beta = 1$.

In this paper we consider simple patterns that appear in $\mathcal{W}$. For example, consider the set, S_x, of indices s such that $\mathcal{W}(s) = b$ and $\mathcal{W}(s+x) = a$ for some fixed x. We present tools that enable us to write explicit formulas for such sets, and specifically use them to study S_x.

Below we introduce an infinite number of combinatorial games, which we dub Beatty games, which constitute the motivation to study the set S_x.

1.2. Beatty games

In his paper from 1907 (see [21]), Wythoff describes a two-player game played on two piles of tokens, based on the well known game Nim: In each turn, a player is allowed to remove any positive amount of tokens from a single pile (Nim move)

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or the same amount of tokens from both piles (diagonal move). The player making the last move wins. For the uninitiated, we point to the following three papers, from among the extensive literature on Wythoff Nim: [5,8,22].

In [13] and in [8] the authors suggest a natural generalization of this game, called k-Wythoff Nim: A player is allowed, in addition to the Nim moves, to remove x tokens from one pile and y tokens from the other, provided that $|x-y|<k$, where $k \in \mathbb{Z}_{\geq 1}$ is a parameter of the game.

A well known result of Combinatorial Game Theory[1] states that in every finite game, every position is either an N-position – a position from which the **N**ext player can win, no matter what the opponent does, or a P-position –a position from which the **P**revious player can win. The set of P-positions is denoted $\mathcal{P}$.

The two papers [13] and [8] show that for k-Wythoff Nim, the set of P-positions is given by $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where $\alpha = (2-k+\sqrt{k^2+4})/2$ and $\beta = 1 + 1/(\alpha - 1) = \alpha + k$.

A game in which every move can be played from any position (as long as the number of tokens is never negative) is called an *invariant* game. k-Wythoff Nim is an example of such a game. In [7] it was conjectured that for *all* irrationals $1 < \alpha < 2 < \beta$ such that $1/\alpha + 1/\beta = 1$, there exists an invariant game for which the set of P-positions is given by $\mathcal{P} = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$. The conjecture was proven in [16] using the $\star$-operator: This operator takes an invariant game to the invariant game in which the moves are the P-positions of the original game (except for $(0,0)$). The authors prove that applying the $\star$-operator to a game whose *moves* are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 1}\}$ results in the desired invariant game. For more information about the $\star$-operator, see [15]. So here is an infinity of games whose P-positions are known, but their rulesets are not given by a simple explicit formula! We dub all games whose set of P-position is of this form *Beatty games*. Note that from the previous paragraph, it follows that k-Wythoff Nim is a Beatty game.

The question of preservation of $\mathcal{P}$ under the operation of adding invariant moves to Wythoff Nim ($k=1$) was introduced in [6]. The authors provided an efficient algorithm which determines whether adding a specific move of the form: take x tokens from one pile and y from the other, to the game changes $\mathcal{P}$.

The present study generalizes this question to all Beatty games. We will see in Section 7 that we can use the analysis of the set S_x to obtain formulas for the pattern of those subtractions that are "forbidden" because adding them changes $\mathcal{P}$. In fact, the main advance of this paper over [6] is twofold: (i) Extension from the special case of the golden ratio to any irrational in $(1,2)$. (ii) Given integers x, y, [6] provided a polynomial algorithm to decide whether (x, y) can be added as an invariant move to Wythoff Nim. In the present paper, the pattern of all y's for a fixed x is determined polynomially as shown in Section 6.3.

1.3. Outline

The paper is structured as follows:

Section 2 gives two different definitions of $\mathcal{W}[\alpha]$: the first is the definition that appears at the beginning of this introduction, and the second is based on morphisms of words. We prove their equivalence, and then list basic properties of $\mathcal{W}[\alpha]$.

In order to derive an explicit formula for S_x, we will need to define some tools useful for the analysis of $\mathcal{W}[\alpha]$ — this will be done in Section 3. Specifically, consider the partition of the natural numbers, based on the positions of a and b in $\mathcal{W}[\alpha]$. We will see that, in some sense, not all the a's and not all the b's are "the same". This will lead to another partition, finer than the previous one. In fact, we will get an infinite sequence of partitions, each finer than its predecessor. Proposition 3 will give an explicit formula for the sets in each of these partitions.

Sections 4–6 show how to write S_x as a finite union of sets from these partitions. Together with the explicit formula for these sets, we obtain an explicit formula for S_x, and an efficient algorithm (Section 6.3), that given x, outputs this formula. In fact, S_x will turn out to be

$$\dot{\bigcup}_{i=1}^{k} \{A_i \lfloor \gamma_i n \rfloor + B_i n + C_i : n \in \mathbb{Z}_{\geq 1}\} \tag{1}$$

for some $k \in \mathbb{Z}_{\geq 1}$, $A_i, B_i, C_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{R}$ (the algorithm will output these values for any given x), where $\dot{\cup}$ denotes disjoint union. To attain this representation, we employ four steps:

1. Solve a simpler problem for specific values of x (Section 4.1).
2. Generalize to an arbitrary x (Section 4.2).
3. Find a formula for S_x, similar to (1), except for the fact that the disjoint union is replaced by a symmetric difference (Section 5).
4. Finally, convert the symmetric difference to a disjoint union (Section 6).

Sections 7 and 8 contain an application of the study of S_x, to the preservation of P-positions in Beatty games under the operation of adding moves.

[1] For general references on Combinatorial Game Theory see, for example, [4,3,1] and [20].