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Thickness and outerthickness for embedded graphs

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ABSTRACT

We prove new upper bounds for the thickness and outerthickness of a graph in terms of its orientable and nonorientable genus by applying the method of deleting spanning disks of embeddings to approximate the thickness and outerthickness. We also show that every non-planar toroidal graph can be edge partitioned into a planar graph and an outerplanar graph. This implies that the outerthickness of the torus (the maximum outerthickness of all toroidal graphs) is 3. Finally, we show that all graphs embeddable in the double torus have thickness at most 3 and outerthickness at most 5.

1. Introduction and terminology

An *outerplanar graph* is a planar graph that can be embedded in the plane without crossing edges, in such a way that all the vertices are incident with the same face. The *thickness* of a graph G, denoted by $\theta(G)$ (first defined by Tutte [19]), is the minimum number of planar subgraphs whose union is G. Similarly, the *outerthickness* $\theta_o(G)$ is obtained when "planar subgraphs" is replaced by "outerplanar subgraphs" in the previous definition. If Σ is a surface, define $\theta(\Sigma) = \max\{\theta(G) : G$ is embeddable in $\Sigma\}$, where the maximum is taken over all graphs embeddable in Σ. Define $\theta_o(\Sigma)$ analogously.

Much work has been done in partitioning the edges of graphs such that each subset induces a subgraph of a certain type. A well-known result by Nash–Williams [17] gives a necessary and sufficient condition for a graph to admit an edge-partition into a fixed number of forests. His results show that any planar graph can be edge-partitioned into three forests, and any outerplanar graph into two forests. Much research has been devoted to partitioning the edges into planar graphs (to determine the thickness of graphs) and outerplanar graphs (to determine the outerthickness of graphs). The thickness of some special classes of graphs has been determined, including the complete graphs K_n [1,20] (see (1.1)), the complete bipartite graphs $K_{m,n}$ [3] (except possibly if m and n are both odd, or $m \leq n$ and n takes some special values), and the hypercube Q_n [13]. See the survey paper [16] for more results on the thickness of graphs. Guy and Nowakowski [9,10] determined the outerthickness of complete graphs (see (1.2)), the hypercube and some complete bipartite graphs. For complete graphs, the results are

$$\theta(K_n) = \begin{cases} \lfloor \frac{n+7}{6} \rfloor, & \text{if } n \neq 9, 10, \\ 3, & \text{otherwise}, \end{cases} \tag{1.1}$$

and

$$\theta_o(K_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil, & \text{if } n \neq 7, \\ 3, & n = 7. \end{cases} \tag{1.2}$$

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It is known that the thickness problem is $\mathcal{NP}$-hard [14]. For many other classes of graphs, attention has been focused on finding upper bounds of thickness and outerthickness. Jünger et al. [11] have shown that a graph has thickness at most 2 if it contains no K_5-minor. Asano [2] proved that, if a graph G is triangle free and has orientable genus γ, then $\theta(G) \leq \gamma(G)+1$. He also showed that all toroidal graphs have thickness at most 2. Dean and Hutchinson [5] strengthened Asano's result by proving that $\theta(G) \leq 6 + \sqrt{2\gamma(G) - 1}$.

In 1971, Chartrand, Geller and Hedetniemi [4] conjectured that every planar graph has an edge partition into two outerplanar graphs. Ding, Oporowski, Sanders and Vertigan [6] proved that every planar graph has an edge partition into two outerplanar graphs and a *vee-forest*, where a vee-forest is the disjoint union of a number of K_2's and $K_{1,2}$'s. They also showed that every graph with nonnegative Euler characteristic has an edge partition into two graphs of tree-width at most three. Kedlaya [12] showed that some planar graphs cannot be edge-partitioned into two outerplanar subgraphs such that one of them is outerplanarly embedded. In 2005 Gonçalves [7,8] announced that he had solved the Chartrand, Geller and Hedetniemi's conjecture.

In this paper, we first provide some technical results in Section 2. We introduce the technique of deleting maximal spanning disks for embeddings of graphs. We also introduce essential edges with respect to spanning disks of embeddings, and the corresponding noncontractible nonhomotopic loop system of surfaces. Applying these techniques we provide results on thickness and outerthickness to (i) improve Dean and Hutchinson's upper bounds for graphs in terms of their orientable and nonorientable genus (Section 3), and (ii) obtain upper bounds for outerthickness of graphs in terms of their orientable and nonorientable genus (Section 3). Moreover we prove that (iii) every non-planar toroidal graph can be edge partitioned into a planar graph and an outerplanar graph. We (iv) improve Asono's result by dropping his triangle free condition (Section 4), and (v) obtain upper bounds of thickness and outerthickness for graphs embeddable in double torus and triple torus (Section 4).

2. Technical results

We prove some technical and structural results in this section. The following is obvious.

Lemma 2.1. *If G is a subgraph of H, then $\theta(G) \leq \theta(H)$ and $\theta_o(G) \leq \theta_o(H)$.*

Lemma 2.1 may not be true if the subgraph relation is replaced by the minor relation, i.e., if G is a minor of H, it is possible that $\theta(G) > \theta(H)$ and $\theta_o(G) > \theta_o(H)$. This may add difficulty to problems of finding thickness and outerthickness since many techniques and results on the minor relation cannot be applied here.

Since adding multiple edges to a graph G does not increase the thickness/outerthickness of G, and the thickness/outerthickness of a graph is equal to the maximum thickness/outerthickness of its blocks, we may assume that graphs are simple and 2-connected. Let S_g be the orientable surface with genus g (the sphere with g handles, $g \geq 0$) and N_k be the nonorientable surface with nonorientable genus k (the sphere with k crosscaps, $k \geq 1$). Suppose C is a cycle of a graph embedded in surface Σ, and x and y are two vertices on C. We assign a direction to C and define xCy to be the open path from x to y in this direction.

In order to study thickness/outerthickness of graphs embedded in surfaces, we apply Lemma 2.1 by adding edges to the embedding of G to obtain a spanning supergraph H of G then study the thickness/outerthickness of H. In this way we obtain a better structure of embeddings.

Let G be a graph and $\Psi(G)$ be an embedding of G in a surface Σ. A subembedding Ψ^s is *spanning* if it contains all vertices of G. A spanning subembedding is *contractible* if it does not contain any noncontractible cycle of $\Psi(G)$. In particular a contractible spanning subembedding is a *spanning disk* if it is homeomorphic to a closed disk, in which case the boundary of this spanning subembedding is a contractible cycle of $\Psi(G)$. For any embedding, a spanning tree is always a contractible spanning subembedding. However, an embedding needs not contain a spanning disk. An example is the unique embedding of the Heawood graph in the torus which is the dual embedding of K_7. It contains no spanning disk even though the embedding is 3-representative (or equivalently, a polyhedral embedding, or a wheel-neighborhood embedding). An edge e is *essential*, with respect to a contractible spanning subembedding Ψ^s if $e \cup \Psi^s$ contains a noncontractible cycle. Note that if e is an essential edge then e is contained in every noncontractible cycle of $e \cup \Psi^s$. An essential edge becomes a noncontractible loop if Ψ^s is contracted to a single point.

Lemma 2.2. *Let G be a simple graph and $\Psi(G)$ be an orientable genus embedding or a minimal surface embedding (i.e., with maximum Euler characteristic) of G in Σ. Then either $\Psi(G)$ contains a spanning disk, or there is a supergraph H with embedding $\Psi(H)$ in Σ such that H is simple, $V(H) = V(G)$, $\Psi(G)$ is a subembedding of $\Psi(H)$, and $\Psi(H)$ contains a spanning disk.*

Proof. Let $\Psi(G)$ be an orientable genus embedding or a minimal surface embedding of G in Σ. Clearly, any embedding in Σ containing $\Psi(G)$ as a subembedding is an orientable genus embedding or a minimal surface embedding. We start with a spanning tree T of G, and add more faces to T such that the resulting subembedding R is maximal (with as many faces as possible) and contractible (but may not be homeomorphic to a disk, e.g., with cut vertices). We then either add new edges to R one by one or change the embedding by re-embedding some edges such that the resulting graph is a supergraph G^+ of G with the same vertex set, and has a spanning region R^+ which is contractible. We do not add multiple edges to G, and thus