Discrete Mathematics 341 (2018) 1210–1214

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Characterizing the number of coloured m-ary partitions modulo m, with and without gaps

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ARTICLE INFO

Article history:
Received 25 January 2017
Received in revised form 9 January 2018
Accepted 19 January 2018

Keywords:
Partition
Congruence
Generating function

ABSTRACT

In a pair of recent papers, Andrews, Fraenkel and Sellers provide a complete characterization for the number of m-ary partitions modulo m, with and without gaps. In this paper we extend these results to the case of coloured m-ary partitions, with and without gaps. Our method of proof is different, giving explicit expansions for the generating functions modulo m.

1. Introduction

An m-ary partition is an integer partition in which each part is a nonnegative integer power of a fixed integer $m \geq 2$. An m-ary partition *without gaps* is an m-ary partition in which m^j must occur as a part whenever m^{j+1} occurs as a part, for every nonnegative integer j.

Recently, Andrews, Fraenkel and Sellers [2] found an explicit expression that characterizes the number of m-ary partitions of a nonnegative integer n modulo m; remarkably, this expression depended only on the coefficients in the base m representation of n. Subsequently Andrews, Fraenkel and Sellers [3] followed this up with a similar result for the number of m-ary partitions without gaps, of a nonnegative integer n modulo m; again, they were able to obtain a (more complicated) explicit expression, and again this expression depended only on the coefficients in the base m representation of n. See also Edgar [6] and Ekhad and Zeilberger [7] for more on these results.

The study of congruences for integer partition numbers has a long history, starting with the work of Ramanujan (see, e.g., [8]). For the special case of m-ary partitions, a number of authors have studied congruence properties, including Churchhouse [5] for $m = 2$, Rødseth [9] for m a prime, and Andrews [1] for arbitrary positive integers $m \geq 2$. The numbers of m-ary partitions without gaps had been previously considered by Bessenrodt, Olsson and Sellers [4] for $m = 2$.

In this note, we consider m-ary partitions, with and without gaps, in which the parts are *coloured*. To specify the number of colours for parts of each size, we let $\mathbf{k} = (k_0, k_1, \ldots)$ for positive integers $k_0, k_1, \ldots$, and say that an m-ary partition is $\mathbf{k}$-*coloured* when there are k_j colours for the part m^j, for $j \geq 0$. This means that there are k_j different kinds of parts of the same size m^j. Let $b_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n, and let $c_m^{(\mathbf{k})}(n)$ denote the number of $\mathbf{k}$-coloured m-ary partitions of n without gaps. For the latter, some part m^j of any colour must occur as a part whenever some part m^{j+1} of any colour (not necessarily the same colour) occurs as a part, for every nonnegative integer j. (In the special case that $k_j = k$ for all $j \geq 0$, where k is a positive integer, we say that the m-ary partitions are k-*coloured*.)

We extend the results of Andrews, Fraenkel and Sellers in [2] and [3] to the case of $\mathbf{k}$-coloured m-ary partitions, where m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$. Our method of proof is different, giving explicit expansions for the generating

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https://doi.org/10.1016/j.disc.2018.01.017
0012-365X/

functions modulo m. We then extract the coefficients in these generating functions to determine explicit expressions for the corresponding numbers of partitions modulo m, stated in the following pair of results.

Theorem 1.1. *For $n \geq 0$, suppose that the base m representation of n is given by*

$$n = d_0 + d_1 m + \cdots + d_t m^t, \qquad 0 \leq t.$$

If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$b_m^{(\mathbf{k})}(n) \equiv \binom{k_0 - 1 + d_0}{k_0 - 1} \prod_{j=1}^{t} \binom{k_j + d_j}{k_j} \pmod{m}.$$

Theorem 1.2. *For $n \geq 1$, suppose that n is divisible by m, with base m representation given by*

$$n = d_s m^s + \cdots + d_t m^t, \qquad 1 \leq s \leq t,$$

where $1 \leq d_s \leq m-1$, and $0 \leq d_{s+1}, \ldots, d_t \leq m-1$. If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then for $0 \leq d_0 \leq m-1$ we have

$$c_m^{(\mathbf{k})}(n - d_0) \equiv \binom{k_0 - 1 - d_0}{k_0 - 1} \left(\varepsilon_s + (-1)^{s-1} \left\{ \binom{k_s + d_s - 1}{k_s} - 1 \right\} \sum_{i=s}^{t} \prod_{j=s+1}^{i} \left\{ \binom{k_j + d_j}{k_j} - 1 \right\} \right) \pmod{m},$$

where $\varepsilon_s = 0$ if s is even, and $\varepsilon_s = 1$ if s is odd.

Theorem 1.1 is proved in Section 2, and Theorem 1.2 is proved in Section 3.

2. Coloured *m*-ary partitions

In this section we consider the following generating function for the numbers $b_m^{(\mathbf{k})}(n)$ of $\mathbf{k}$-coloured m-ary partitions:

$$B_m^{(\mathbf{k})}(q) = \sum_{n=0}^{\infty} b_m^{(\mathbf{k})}(n) q^n = \prod_{j=0}^{\infty} \left(1 - q^{m^j}\right)^{-k_j}.$$

The following simple result will be key to the expansion of $B_m^{(\mathbf{k})}(q)$ modulo m.

Proposition 2.1. *For positive integers m, a with m relatively prime to $(a-1)!$, we have*

$$(1-q)^{-a} \equiv \left(1 - q^m\right)^{-1} \sum_{\ell=0}^{m-1} \binom{a - 1 + \ell}{a - 1} q^{\ell} \pmod{m}.$$

Proof. From the binomial theorem we have

$$(1-q)^{-a} = \sum_{\ell=0}^{\infty} \binom{a-1+\ell}{a-1} q^{\ell}.$$

Now using the falling factorial notation $(a-1+\ell)_{a-1} = (a-1+\ell)(a-2+\ell)\cdots(1+\ell)$ we have

$$\binom{a-1+\ell}{a-1} = ((a-1)!)^{-1} (a-1+\ell)_{a-1}.$$

But

$$(a-1+\ell+m)_{a-1} \equiv (a-1+\ell)_{a-1} \pmod{m},$$

for any integer ℓ, and $((a-1)!)^{-1}$ exists in $\mathbb{Z}_m$ since m is relatively prime to $(a-1)!$, which gives

$$\binom{a-1+\ell+m}{a-1} \equiv \binom{a-1+\ell}{a-1} \pmod{m}, \tag{1}$$

and the result follows. □

We are now able to give an explicit expansion for $B_m^{(\mathbf{k})}(q)$ modulo m.

Theorem 2.2. *If m is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have*

$$B_m^{(\mathbf{k})}(q) \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \prod_{j=1}^{\infty} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j} \right) \pmod{m}.$$