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Graphical characterization of positive definite non symmetric quasi-Cartan matrices

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ABSTRACT

It is known that each positive definite quasi-Cartan matrix A is $\mathbb{Z}$-equivalent to a Cartan matrix A_Δ called Dynkin type of A, the matrix A_Δ is uniquely determined up to conjugation by permutation matrices. However, in most of the cases, it is not possible to determine the Dynkin type of a given connected quasi-Cartan matrix by simple inspection. In this paper, we give a graph theoretical characterization of non-symmetric connected quasi-Cartan matrices. For this purpose, a special assemblage of blocks is introduced. This result complements the approach proposed by Barot (1999, 2001), for $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_m$ with $m = 6, 7, 8$.

1. Introduction and results

Quasi-Cartan matrices are present in many areas of mathematics. The motivation is based on the classical theory of complex semi-simple Lie algebras, (see [12]). These algebras can be characterized by a base of the root system from which a Cartan matrix is obtained. A *symmetrizer* of a matrix A is an integer diagonal matrix D with positive diagonal entries such that DA is symmetric. If A has a symmetrizer D then A is called *symmetrizable* (D is not unique). Following [6], by a *quasi-Cartan* matrix of size $n \geq 2$ we mean a square $n \times n$ matrix $A = [A_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ with integer coefficients A_{ij} such that A is symmetrizable and $A_{ii} = 2$, for all i. The set of all quasi Cartan matrices $A \in \mathbb{M}_n(\mathbb{Z})$ is denoted by $\mathbf{qC}$. We say that a matrix $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ is *positive definite*, if the symmetric matrix $DA \in \mathbb{M}_n(\mathbb{Z}) \subset \mathbb{M}_n(\mathbb{R})$ is positive definite, for some symmetrizer D. The set of all positive definite quasi-Cartan matrices $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ is denoted by $\mathbf{qC}^+$. We note that a matrix $A \in \mathbf{qC}^+$ is a Cartan matrix if $A_{ij} \leq 0$ for all pairs i, j with $i \neq j$. The quasi-Cartan matrices $A, A' \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ are defined to be $\mathbb{Z}$-*equivalent* (we denote it by $A \sim A'$) if there exists a $\mathbb{Z}$-invertible matrix $E \in \mathbb{M}_n(\mathbb{Z})$ and symmetrizers $D, D' \in \mathbb{M}_n(\mathbb{Z})$ such that $D'A' = E^t(DA)E$ and D' is conjugate to D by a permutation matrix. For general purposes, it will be convenient to switch to a more graphical language.

Following [7], by a *mixed graph* we mean the triple $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that consists of a set $\mathcal{V} \neq \emptyset$ of vertices, a set $\mathcal{E}$ of edges (undirected) and a set $\mathcal{A}$ of arrows. In this paper, a *bigraph* B is a mixed graph G together with a function $\omega : \mathcal{E} \cup \mathcal{A} \to \mathbb{Z}$ that assigns to every $e \in \mathcal{E} \cup \mathcal{A}$ an integer number called the *weight* of e. A vertex $v \in \mathcal{V}$ is a *source* (respectively *sink*) vertex if for all $a_{ij} \in \mathcal{A}$ the vertex i (respectively j) is equal to v and $e_{vj}, e_{iv} \notin \mathcal{E}$.

Definition 1.1. To each quasi-Cartan matrix $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$, with $n \geq 2$, we associate its bigraph $B_A = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \omega)$, with n vertices, as follows:

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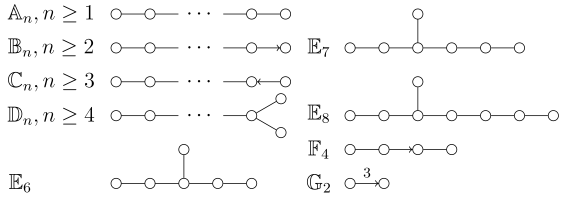


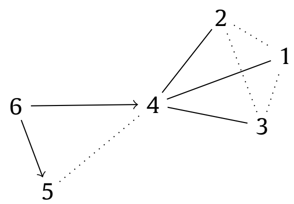
Fig. 1. Dynkin diagrams.

- $\mathcal{V} = \{1, 2, \ldots, n\}$
- $\mathcal{E} = \{e_{ij} \mid i, j \in \mathcal{V}$ with $i \neq j$ and $|A_{ij}| = |A_{ji}| \neq 0\}$
- $\mathcal{A} = \{a_{ij} \mid i, j \in \mathcal{V}$ with $i \neq j$ and $|A_{ij}| < |A_{ji}|\}$
- for each $e \in \mathcal{E} \cup \mathcal{A}$, we set $\omega(e) = A_{ji}$, where $|A_{ij}| \leq |A_{ji}|$.

Notice that from Definition 1.1 the sets $\mathcal{E}$ and $\mathcal{A}$ are disjoint. A *path* P in B_A from vertex v_1 to vertex v_r is a subgraph $P = v_1 v_2 \ldots v_r$ induced in B_A by the set of vertices $v_i \in \mathcal{V}$ where for all i, $1 \leq i \leq r$ the vertices v_i are pairwise distinct and there exists $e \in \mathcal{E} \cup \mathcal{A}$ between the vertices v_i and v_{i+1}. We say that B_A is *connected* if there exists a path from v_i to v_j for all $v_i, v_j \in \mathcal{V}$ [7]. A *chordless cycle* is a connected induced sub-bigraph such that every vertex is adjacent with exactly two vertices. A bigraph B_A satisfies the *chordless cycle condition* if every induced chordless cycle of B_A has an odd number of dotted connections (edges or arrows). Every bigraph B_A associated to a quasi-Cartan matrix A can be represented as a diagram of dots (vertices in B_A), lines and arrows (solid and dotted). All edges and arrows are represented as follows: if $e_{ij} \in \mathcal{E}$ then e_{ij} is indicated by a dotted line with weight ω, $i \overset{\omega}{\cdots} j$ if $\omega(e_{ij}) > 0$, and solid $i \overset{\omega}{—} j$ if $\omega(e_{ij}) < 0$. Similarly for $a_{ij} \in \mathcal{A}$, a_{ij} is indicated by a dotted arrow $i \overset{\omega}{\dashrightarrow} j$ if $\omega(a_{ij}) > 0$, and solid $i \overset{\omega}{\longrightarrow} j$ if $\omega(a_{ij}) < 0$. We denote by $\Phi(A)$ the *frame* of a quasi-Cartan matrix A, that is, the graph obtained from B_A by turning all broken edges and broken arrows into solid ones [3]. A frame $\Phi(A)$ is called *positive* if A is a positive definite matrix. Throughout this paper, all the solid (dotted) arrows are considered with $\omega = -2$ ($\omega = 2$) unless otherwise indicated, and no distinction is made between the bigraph B_A and its diagram.

Example 1.2. A quasi-Cartan matrix and its associated bigraph.

$$\begin{pmatrix} 2 & 1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$



If A is a Cartan matrix, the bigraph B_A is actually a bigraph with $\omega(e) < 0$ for all $e \in \mathcal{E} \cup \mathcal{A}$, moreover if A is connected (i.e. B_A is connected) then B_A is known as *Dynkin diagram* (see Fig. 1). From now on, we will only consider connected matrices.

If $A' \in \mathbb{M}_n(\mathbb{Z})$ is a connected quasi-Cartan in $\mathbf{qC}^+$, and A_Δ is a Cartan matrix such that $A' \sim A_\Delta$, then Δ will be referred to be the *Dynkin type* of B_A, that is, the Dynkin diagram associated to A_Δ . The existence of the Cartan matrix A_Δ such that $A' \sim A_\Delta$ will be proved in the Section 2, see also [13], a proof for the symmetric case is given in [8]. It follows that two connected matrices in $\mathbf{qC}^+$ with the same Dynkin type are $\mathbb{Z}$-equivalent; therefore, it is important to have a simple characterization of positive definite connected quasi-Cartan matrices. For this purpose we study in the following paragraph some graphical and combinatorial aspects for the various parameters characterizing the Dynkin types of positive definite connected quasi-Cartan matrices.

Let X and Y be disjoint sets of vertices. We denote by $F[X, Y]$ the non-separable bigraph obtained by joining each pair of vertices x, y with $x \in X$ and $y \in Y$ by a solid edge, and all other pairs of vertices by a dotted edge; such bigraph is called an $\mathbb{A}$-*block*, see [2], [1]. If v is a vertex in $F[X, Y]$ and $|X \cup Y| \geq 2$, we denote by $\overrightarrow{F_v}[X, Y]$ ($\overleftarrow{F_v}[X, Y]$) and we call $\mathbb{B}$-block ($\mathbb{C}$-block) to the bigraph obtained from $F[X, Y]$ after substituting every solid or dotted edge over v by a solid or dotted arrow pointing to (coming out of) the vertex v. The vertex v is the sink (source) vertex of $\overrightarrow{F_v}[X, Y]$ ($\overleftarrow{F_v}[X, Y]$). In both cases, we call to vertex v a *distinguished vertex*. (See Fig. 2.)

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \omega)$, $G' = (\mathcal{V}', \mathcal{E}', \mathcal{A}', \omega')$. Then, we define the sum of G and G' by $G \oplus G' = (\mathcal{V} \cup \mathcal{V}', \mathcal{E}'', \mathcal{A}'', \omega'')$ where:

$$\omega''(e) = \begin{cases} \omega(e), & \text{if } e \in (\mathcal{E} \setminus \mathcal{E}') \cup (\mathcal{A} \setminus \mathcal{A}') \\ \omega'(e), & \text{if } e \in (\mathcal{E}' \setminus \mathcal{E}) \cup (\mathcal{A}' \setminus \mathcal{A}) \\ \omega'(e) + \omega(e), & \text{if } e \in (\mathcal{E} \cap \mathcal{E}') \cup (\mathcal{A} \cap \mathcal{A}') \end{cases}$$

$\mathcal{E}'' = (\mathcal{E} \cup \mathcal{E}') \setminus \{e \in \mathcal{E} \cap \mathcal{E}' \mid \omega'(e) + \omega(e) = 0\}$ and $\mathcal{A}'' = (\mathcal{A} \cup \mathcal{A}') \setminus \{e \in \mathcal{A} \cap \mathcal{A}' \mid \omega'(e) + \omega(e) = 0\}$.