Discrete Mathematics 341 (2018) 1708–1721

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Recolouring reflexive digraphs

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ARTICLE INFO

*Article history:*
Received 16 August 2017
Received in revised form 4 March 2018
Accepted 5 March 2018
Available online 31 March 2018

*Keywords:*
Homomorphisms
Recolouring
Reflexive digraphs
Hom-graph

ABSTRACT

Given digraphs $G$ and $H$, the colouring graph $\mathbf{Col}(G, H)$ has as its vertices all homomorphism of $G$ to $H$. There is an arc $\phi \to \phi'$ between two homomorphisms if they differ on exactly one vertex $v$, and if $v$ has a loop we also require $\phi(v) \to \phi'(v)$. The *recolouring problem* asks if there is a path in $\mathbf{Col}(G, H)$ between given homomorphisms $\phi$ and $\psi$. We examine this problem in the case where $G$ is a digraph and $H$ is a reflexive, digraph cycle.

We show that for a reflexive digraph cycle $H$ and a reflexive digraph $G$, the problem of determining whether there is a path between two maps in $\mathbf{Col}(G, H)$ can be solved in time polynomial in $G$. When $G$ is not reflexive, we show the same except for certain digraph 4-cycles $H$.



## 1. Introduction

For (undirected, irreflexive) graphs $G$ and $H$, the *colour graph* $\mathbf{Col}(G, H)$ is the graph whose vertex set is the set $\mathrm{Hom}(G, H)$ of homomorphisms from $G$ to $H$, also known as $H$-colourings of $G$, and in which there is an edge $\phi \sim \psi$ between homomorphisms $\phi, \psi \in \mathrm{Hom}(G, H)$ if they differ on exactly one vertex of $G$. $\mathbf{Col}(G, H)$ is one of several constructions in the literature for applying topological ideas to graph theoretical problems. A walk between vertices of $\mathbf{Col}(G, H)$ is a strong analogue of the notion of a homotopy between continuous maps of one space $G$ into another space $H$. So connectivity in $\mathbf{Col}(G, H)$ is of interest, and has been investigated by many authors.

The decision problem $\textsc{Recol}(H)$ takes as a instance a graph $G$ and two $H$-colourings of $G$. The task is to decide if there is a walk in $\mathbf{Col}(G, H)$ between the $H$-colourings.

The papers [1,7–9] address the problem of $\textsc{Recol}(K_k)$, and the closely related problems such as deciding, for an instance $G$, whether or not $\mathbf{Col}(G, K_k)$ is connected, or whether or not it has a Hamilton cycle. From these papers, we get, in particular, that $\textsc{Recol}(K_k)$ is polynomial-time solvable for $k \leq 3$ [8] and PSPACE-complete for $k \geq 4$ [1]. In fact, the problem remains PSPACE-complete when the input is restricted to bipartite instances for $k \geq 4$, to planar instances for $4 \leq k \leq 6$, or to bipartite planar instances for $k = 4$.

Moving from colourings to circular colourings, similar topics were addressed for the circular cliques $G_{p,q}$ in [3–5]. In particular, in [3] it is shown that $\textsc{Recol}(G_{p,q})$ is polynomial-time solvable if $p/q < 4$ and is PSPACE-complete if $p/q \geq 4$. More recently, Wrochna [13] proved a strikingly general result. He showed that $\textsc{Recol}(H)$ is polynomial-time solvable when $H$ is $C_4$-free.

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https://doi.org/10.1016/j.disc.2018.03.006
0012-365X/

### 1.1. Reflexive digraphs

In this paper, we consider finite digraphs. We allow loops and symmetric arc pairs (simply called symmetric edges). A digraph is *reflexive* (resp. *irreflexive*) if all vertices (resp. no vertex) have a loop. If $(u, v)$ is an arc of a digraph $G$ we write $u \to v$. Varying from standard notation, we refer to an **ordered** pair $uv$ of vertices of $G$ as an *edge*, and write $u \sim v$ if either $u \to v$ or $u \leftarrow v$ (or both). So $u \sim v$ if $v \sim u$. When both $u \to v$ and $u \leftarrow v$ hold, we often write $u \leftrightarrow v$. A *walk* $W$ from $w_1$ to $w_\ell$ is a sequence $w_1 \sim w_2 \sim \cdots \sim w_\ell$ of consecutively adjacent vertices. It is *directed* if $w_1 \to w_2 \to \cdots \to w_\ell$ and *symmetric* if $w_1 \leftrightarrow w_2 \leftrightarrow \cdots \leftrightarrow w_\ell$.

As we mentioned above, $\mathbf{Col}(G, H)$ is just one of several structures for applying topological ideas to graphs. To consider how to extend this to reflexive digraphs, we look at another, more well-known, such construction: the Hom-graph.

Recall that a *homomorphism* $\phi : G \to H$ of digraphs $G$ and $H$ is a (not necessarily injective) vertex map $\phi : V(G) \to V(H)$, such that $\phi(u) \to \phi(v)$ in $H$ whenever $u \to v$ in $G$. The *Hom-graph* $\mathbf{Hom}(G, H)$ is a digraph on the vertex set $\mathrm{Hom}(G, H)$ of homomorphisms from $G$ to $H$. For homomorphisms $\phi$ and $\phi'$ of $\mathrm{Hom}(G, H)$, $\phi \to \phi'$ is an arc of $\mathbf{Hom}(G, H)$ if $\phi(u) \to \phi'(v)$ in $H$ whenever $u \to v$ in $G$.

Viewing a graph as a symmetric irreflexive digraph, it is not hard to see that $\mathbf{Col}(G, H)$ is simply the subgraph of $\mathbf{Hom}(G, H)$ consisting of edges whose endpoints differ on exactly one vertex. For reflexive digraphs, we therefore define $\mathbf{Col}(G, H)$ as the subgraph of $\mathbf{Hom}(G, H)$ consisting of arcs whose endpoints differ on exactly one vertex.

Our main interest is the following problem for a fixed digraph $H$.
Recol($H$)

| | |
|---|---|
| Instance: | $(G, \phi, \psi)$: a digraph $G$, and two homomorphisms $\phi, \psi \in \mathrm{Hom}(G, H)$. |
| Question: | Is there a walk from $\phi$ to $\psi$ in $\mathbf{Col}(G, H)$? |

While loops on $G$ have no effect on the set $\mathrm{Hom}(G, H)$, they do effect the arcs of $\mathbf{Col}(G, H)$. Indeed, if a vertex $v$ of $G$ has a loops then for an arc $\phi \to \phi'$ in $\mathbf{Col}(G, H)$ we necessarily have that $\phi(v) \to \phi'(v)$, while this is not necessary if $v$ has no loop. We let Recol$_r(H)$ denote the problem Recol($H$) restricted to reflexive instances.

### 1.2. Results

A *(digraph) cycle* $C = v_1 \sim v_2 \sim \cdots \sim v_n \sim v_1$ of girth $n$ is a closed walk with $n$ distinct vertices. An edge $v_i \sim v_{i+1}$ of $C$ is *forward* if $v_i \to v_{i+1}$ and $v_i \not\leftarrow v_{i+1}$, *backward* if $v_i \leftarrow v_{i+1}$ and $v_i \not\to v_{i+1}$, and *symmetric* if $v_i \leftrightarrow v_{i+1}$. A cycle is *symmetric* or *directed* if it is as a walk. It is *oriented* if no edge is symmetric. The *algebraic girth* of an oriented cycle is the number of forward edges minus the number of backwards edges.

**Theorem 1.1.** *For a reflexive digraph cycle $B$ the problem* Recol$_r(B)$ *is polynomial-time solvable, and unless $B$ contains a 4-cycle of algebraic girth 0,* Recol($B$) *is also polynomial-time solvable.*

Observe that, up to isomorphism, there are two oriented 4-cycles of algebraic girth 0, and ten 4-cycles that contain one of them. In the next section we make some simple reductions to this theorem and explain the exceptional cases; but before that, we look at couple more very natural variants on Recol($H$) that come out of different notions of extending the problem to digraphs $H$.

One can ask when there is a directed or symmetric walk from $\phi$ to $\psi$ in $\mathbf{Col}(G, H)$. These variants were in fact where our investigations began, but it turns out that the results we get about them follow, with only small work, from our proofs about general walks; so we deal with them only briefly in Section 8.

For undirected, irreflexive graphs, there is a path between maps in $\mathbf{Hom}(G, H)$ if and only if there is path between them in $\mathbf{Col}(G, H)$. See for example [5]. This does not hold when $H$ contains non-symmetric edges. Thus, another natural question is asking if there is a walk between $\phi$ and $\psi$ in $\mathbf{Hom}(G, H)$, or in the subgraph $\mathbf{Hom}_i(G, H)$ of edges that differ on at most $i$ vertices. This adds considerable complication, and is studied in our companion paper [2].

## 2. An initial reduction and an outline of the proof

The topological properties of a cycle $B$ will play a large factor in our proof; there are three types of cycles to consider. The first type is small cycles, or 'topologically contractible' cycles for which Recol is trivial. For this we appeal to the literature. Though we are unaware of any papers that address the problem Recol($H$) explicitly for reflexive graphs $H$, there is applicable literature. In [10], the authors investigate the relation between the existence of an *NU*-operation on $H$ and the connectedness of $\mathbf{Hom}(G, H)$, and so of $\mathbf{Col}(G, H)$, for various symmetric graphs $G$. They show, that if a reflexive graph $H$ is dismantlable then $\mathbf{Col}(G, H)$ is connected for all reflexive graphs $G$.

It follows that Recol$_r(H)$ is trivial for all dismantlable reflexive symmetric graphs. This includes such reflexive graphs as chordal graphs, and retracts of products of paths, which tells us that Recol$_r(K_n)$ is trivial for any reflexive clique $K_n$. So is Recol($K_n$), because for a clique $K_n$, loops on $G$ have no effect on arcs of $\mathbf{Col}(G, K_n)$. In particular, we have the following.

**Fact 2.1.** *For $n \in [3]$,* Recol($C_n$) *and* Recol$_r(C_n)$ *are trivial for the reflexive symmetric cycle $C_n$.*