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On the number of single-peaked narcissistic or single-crossing narcissistic preference profiles

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ABSTRACT

We investigate preference profiles for a set $\mathcal{V}$ of voters, where each voter i has a preference order $\succ_i$ on a finite set A of alternatives (that is, a linear order on A) such that for each two alternatives $a, b \in A$, voter i *prefers* a to b if $a \succ_i b$. Such a profile is *narcissistic* if each alternative a is preferred the *most* by at least one voter. It is *single-peaked* if there is a linear order $\rhd^{\text{sp}}$ on the alternatives such that each voter's preferences on the alternatives along the order $\rhd^{\text{sp}}$ are either strictly increasing, or strictly decreasing, or first strictly increasing and then strictly decreasing. It is *single-crossing* if there is a linear order $\rhd^{\text{sc}}$ on the voters such that each pair of alternatives divides the order $\rhd^{\text{sc}}$ into at most two suborders, where in each suborder, all voters have the same linear order on this pair. We show that for n voters and n alternatives, the number of single-peaked narcissistic profiles is $\prod_{i=2}^{n-1}\binom{n-1}{i-1}$ while the number of single-crossing narcissistic profiles is $2^{\binom{n-1}{2}}$.

1. Introduction

We deal with permutations of an n-element set $A := \{1, 2, \ldots, n\}$ that satisfy some specific properties. These properties arise from social choice theory, where each permutation is interpreted as the preference order of an individual on the set A. The elements of A are called *alternatives*. In the following, we will first use terminology established in social choice, and then introduce notions that are more commonly used in discrete mathematics.

Social choice theory, and voting theory in particular, deals with voters and their preferences on a set of alternatives. There, each voter i from a voter set $\mathcal{V}$ has a *preference order* $\succ_i$ *on the set* A (which is a linear order on A), such that for each two alternatives $a, b \in A$, voter i *prefers* a to b if $a \succ_i b$ holds.

When forming coalitions [8,16], building teams or finding partners [5,9,12,13,24], or playing games [25], the individuals, who we jointly denote as *voters*, may have preferences on the alternatives as potential coalition partners, team members, or players. In such situations, the voters and alternatives are identical, that is, $A = \mathcal{V}$. Deriving from a simple psychological model, it seems natural to assume that each voter is *narcissistic* [5], meaning that she is her own ideal and, thus, *most preferred alternative*. In other words, for each voter $i \in \mathcal{V}$ and each alternative $b \in \mathcal{V} \setminus \{i\}$, it holds that $i \succ_i b$.

Another well-studied property of voters preference orders on the set A of alternatives, the *single-peaked* property, is characterized by a linear order $\rhd^{\text{sp}}$ of the alternatives, where for each voter i, her preferences along the order $\rhd^{\text{sp}}$ strictly increase until they reach the *peak* which is her most preferred alternative, and then strictly decrease. In other words, for each alternative $b \in A$, the set $\{b\} \cup \{a \in A \mid a \succ_i b\}$ forms an interval in $\rhd^{\text{sp}}$. Black [6] introduced the concept of

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single-peakedness. He observed that voters' preferences on political parties are single-peaked, meaning that there is a left-to-right political spectrum of the parties such that each voter has a political ideal on this spectrum and the further away a party is from her ideal, the less she will like this party. Single-peaked preferences are also studied in psychology under the name *unimodal orders* [15,17].

A third property, the *single-crossing* property, requires that there is a linear order of the voters such that the preference orders of the voters on each pair of alternatives along this order change at most once. In other words, there is a linear order $\rhd^{sc}$ of the voters where for each two distinct alternatives $a, b \in A$ and for each three distinct voters $i, j, k \in \mathcal{V}$ with $i \rhd^{sc} j \rhd^{sc} k$, if $a \succ_i b$ and $a \succ_k b$, then $a \succ_j b$. Mirrlees [27] and Roberts [28] introduced this concept in the field of economics. They observed that voters' preferences on income taxation display a pattern that depends on their incomes, and are thus single-crossing: When asked about the preferences on two tax rates x and y with $x > y$, if a voter v (the "crossing" spot) with medium income prefers x over y instead of y over x, then all voters with higher income than v will also prefer y over x. Single-crossingness goes back to the work of Karlin [23] and is closely related to a specific partial ordered set on the set of all permutations of a given set, known as the *weak Bruhat order*. We refer to the papers of Abello [1], Bredereck et al. [10] and Galambos and Reiner [22] for more information.

See Sections 2.2–2.4 for a formal definition of the three properties we just discussed.

Research on restricted domains such as single-peaked or single-crossing preferences has been popular in political science, in psychology, in social choice, and quite recently in computational social choice. We refer to the papers of Bredereck et al. [11], Elkind et al. [19] for ample references to research on the two properties. Single-crossing preferences are not necessarily single-peaked, but Saporiti and Tohmé [29] and Barberà and Moreno [4] observed that single-crossing narcissistic preferences are single-peaked. However, not all single-peaked narcissistic preferences are single-crossing. For a simple illustration, the preferences of the following four voters are narcissistic.

$$\begin{array}{lccccccc}
\text{voter } v_1: & v_1 & \succ_1 & v_2 & \succ_1 & v_3 & \succ_1 & v_4, \\
\text{voter } v_2: & v_2 & \succ_2 & v_3 & \succ_2 & v_4 & \succ_2 & v_1, \\
\text{voter } v_3: & v_3 & \succ_3 & v_2 & \succ_3 & v_4 & \succ_3 & v_1, \\
\text{voter } v_4: & v_4 & \succ_4 & v_3 & \succ_4 & v_2 & \succ_4 & v_1.
\end{array}$$

Voter v_1 is her most preferred alternative, and v_2, v_3, and v_4 are voter v_1's second most preferred, third most preferred, and least preferred alternative, respectively. These voter preferences are single-crossing, and also single-peaked, with respect to the order $v_1 \rhd v_2 \rhd v_3 \rhd v_4$.

However, if we just swap the positions of v_4 and v_1 in the preference order of voter v_3 to obtain

$$\begin{array}{lccccccc}
\text{voter } v_3: & v_3 & \succ_3 & v_2 & \succ_3 & v_1 & \succ_3 & v_4,
\end{array}$$

then the resulting voter preferences, together with voters v_1, v_2, and v_4, are still single-peaked (with respect to the order $\rhd$) and narcissistic, but not single-crossing anymore. See Example 2 for further discussion.

In this work, we deal with preference profiles with n voters who each have a preference order on all n voters. In general, there are $n!^n$ different preference profiles. But how likely is it that one of these profiles will have some specific property? For instance, the number of narcissistic profiles is $(n-1)!^n$. So, one out of n^n profiles is narcissistic. Lackner and Lackner [26] studied the likelihood of single-peaked preferences under some distribution assumption on the preference orders of the voters. However, we are interested in narcissistic profiles that are also single-peaked, and that are also single-crossing. More precisely, we investigate the numbers of narcissistic profiles that are also single-peaked (SPN), and of narcissistic profiles that are also single-crossing (SCN), respectively. While it is quite straightforward to derive the number of SPN profiles, this is not the case for SCN profiles. Nonetheless, we are able to determine the number of SCN profiles with the help of *semi-standard Young tableaux* (SSYT), by establishing a bijective relation between SSYTs and SCN profiles.

Our results are that for n voters and n alternatives, the number of single-peaked narcissistic profiles is $\prod_{i=2}^{n-1}\binom{n-1}{i-1}$ while the number of single-crossing narcissistic profiles is $2^{\binom{n-1}{2}}$.

2. Basic definitions and fundamentals

In this section, we introduce basic terms from social choice [2, Chapter 4], combinatorics of permutations [7], and Young tableaux [21,30,31].

2.1. Voters, alternatives, and preference orders

Let $\mathcal{V} := \{1, 2, \ldots, n\}$ be a set of voters. Since we concern ourselves with voters that have preferences over themselves, $\mathcal{V}$ is also the set of alternatives. A *preference order* $\succ$ on $\mathcal{V}$ is a strict linear order on $\mathcal{V}$, that is, a binary relation on $\mathcal{V}$ which is total, antisymmetric, and transitive. Sometimes, we use the letters $a, b, c, \ldots$ instead of the numbers $1, 2, \ldots$ to emphasize that we are considering the alternatives instead of the voters. Given two disjoint subsets of alternatives A and B, we use the notation $A \succ B$ to express that a voter has a preference order $\succ$ such that for each $a \in A$ and for each $b \in B$ it holds that $a \succ b$. We simplify $A \succ B$ to $a \succ B$ if $A = \{a\}$, and $A \succ B$ to $A \succ b$ if $B = \{b\}$.