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Exponential Hilbert series and the Stirling numbers of the second kind

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ABSTRACT

We consider the exponential generating function whose coefficients encode the dimensions of irreducible highest weight representations which lie on a given ray in the dominant chamber of the weight lattice. This formal power series can be considered as an exponential version of the Hilbert series of a flag variety. In this context, we compute a simple closed form for the exponential generating function in terms of finitely many differential operators and the Stirling polynomials. We prove that this series converges to a product of a rational polynomial and an exponential, and that, by summing the constant term and linear coefficient of this polynomial, we recover the dimension of the representation.

1. Introduction

Let G be a semisimple, simply-connected linear algebraic group over $\mathbb{C}$, and fix a choice of Borel subgroup B inside a (fixed) parabolic subgroup P. The flag variety, $X := G/P$, is an object of classical study in both representation theory and invariant theory. In particular, the G-equivariant embeddings of X into a projective space have been classified (see [3,4]). As a projective embedding of a variety induces a natural gradation on its homogeneous coordinate ring, the Hilbert series of the embedding is an object of fundamental interest. For flag varieties, simple closed forms for the Hilbert series are computed in [3]. In [5], the author generalizes the results in [3] to multigradings on the homogeneous coordinate ring, $\mathbb{C}[G/P]$, of G/P and presents a closed form for the multivariate Hilbert series of these multigraded objects. In each of these papers, the authors show that the (singly- or multigraded) Hilbert series can be written in terms of a product of finitely many differential operators.

In this paper, we study the *exponential* Hilbert series of a flag variety. Given a graded algebra A, with homogeneous components A_n ($n \in \mathbb{N}$), we define its *exponential Hilbert series* to be the formal power series

$$E_A(q) := \sum_{n\geq 0} \dim(A_n)\frac{q^n}{n!}. \tag{1.1}$$

While the Hilbert series of an algebra has been studied extensively, $E_A(q)$ has been left basically unconsidered. However, in the context of flag varieties, computations of closed form for $E_A(q)$ yield interesting algebraic and combinatorial results. The series $E_A(q)$ is the exponential generating function whose coefficients encode the dimensions of the graded components of the algebra, A. We recall results from [3,5], as well as present some results on the exponential Hilbert series of a flag variety to be proved later in the paper.

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1.1. Hilbert series of X_λ

Let λ be a dominant integral weight of G. In other words, if we write $P_+(G)$ for the semigroup generated by the fundamental dominant weights of G, then $\lambda \in P_+(G)$. $P_+(G)$ is called the *dominant chamber* of the weight lattice, $P(G)$. The weight, λ, corresponds to a parabolic subgroup, P, of G by letting P be the stabilizer of the unique hyperplane in $L(\lambda)$ fixed by the Borel subgroup, B. Here $L(\lambda)$ denotes the (finite-dimensional) irreducible representation of G with highest weight λ. Then the flag variety $X = G/P$ can be embedded in the projective space $\mathbb{P}(L(\lambda))$ of hyperplanes in $L(\lambda)$ via the map

$$\pi_\lambda : gP \mapsto g.H, \tag{1.2}$$

where H is the hyperplane in question. We denote the image of X under π_λ as X_λ. The coordinate ring, $\mathbb{C}[X_\lambda]$, of this embedding decomposes into (graded) components indexed by an infinite family of highest weight representations of G. Namely,

$$\mathbb{C}[X_\lambda] = \bigoplus_{n\geq 0} L(n\lambda). \tag{1.3}$$

In other words, the graded components of $\mathbb{C}[X_\lambda]$ are exactly the highest weight representations corresponding to the weights of G that lie along the ray $\mathbb{N}\lambda$ in the weight lattice.

Recall that, given a graded algebra, A, with homogeneous components, A_n ($n \in \mathbb{N}$), we can define the *Hilbert series* of A to be the generating function of the dimensions of the homogeneous components. Namely,

$$HS(A) := \sum_{n\geq 0} \dim(A_n)q^n. \tag{1.4}$$

In the case of X_λ, the graded components are irreducible representations of G. The dimensions of these representations can be computed via the *Weyl Dimension Formula* (WDF). Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and B, respectively. The pair $(\mathfrak{g}, \mathfrak{h})$ corresponds to a set of positive roots, Φ^+, on $\mathfrak{g}$. Let $(\cdot,\cdot)$ denote the non-degenerate bilinear form on $\mathfrak{h}^*$ induced by the Killing form. Then the WDF may be stated as follows (see virtually any text on representation theory, for example [2]).

Weyl Dimension Formula. *Let $\lambda \in P_+(G)$. Then*

$$\dim(L(\lambda)) = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)},$$

where $\{\omega_1,\ldots,\omega_k\}$ denote the fundamental dominant weights of $\mathfrak{g}$ and

$$\rho = \sum_{i=1}^{k} \omega_i.$$

Following [3], we define $c_\lambda(\alpha) := (\lambda, \alpha)$. Then the formula given by the WDF simplifies to

$$\dim(L(\lambda)) = \prod_{\alpha\in\Phi^+} (1 + c_\lambda(\alpha)). \tag{1.5}$$

Through a rather sleek argument, the authors of [3] arrive at the following formula for the Hilbert series of X_λ.

Theorem 1. *Let $HS_\lambda(q)$ be the Hilbert series of the embedding X_λ. Then*

$$HS_\lambda(q) = \prod_{\alpha\in\Phi^+} \left(1 + c_\lambda(\alpha)q\frac{d}{dq}\right)\frac{1}{1+q}.$$

Note: in the above formula, the differential operator $\left(1 + c_\lambda(\alpha)q\frac{d}{dq}\right)$ is being applied to the rational function $\frac{1}{1+q}$.

The proof of this theorem can be found in [3].

1.2. Multivariate Hilbert series of X_λ

In [5], the author generalizes the above discussion to *multigradings* on the coordinate rings of flag varieties (by multigrading, we mean an $\mathbb{N}^k$ grading for some $k > 1$). To recall the result, we let $\lambda_1,\ldots,\lambda_k$ be dominant integral weights of G. Note that we do not need to assume that these weights are linearly independent in the discussion to follow. We call the semigroup, $\mathbb{N}\lambda_1 \times \cdots \times \mathbb{N}\lambda_k$, generated by the weights a *lattice cone* in the dominant chamber of the weight lattice. We then consider the series

$$HS_{\mathbf{q}}\langle\lambda_1,\ldots,\lambda_k\rangle := \sum_{\mathbf{a}\in\mathbb{N}^k} \dim(L(a_1\lambda_1+\cdots+a_k\lambda_k))\mathbf{q}^{\mathbf{a}}, \tag{1.6}$$