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Stability in the Erdős–Gallai Theorem on cycles and paths, II[☆]

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ABSTRACT

The Erdős–Gallai Theorem states that for $k \geq 3$, any n-vertex graph with no cycle of length at least k has at most $\frac{1}{2}(k-1)(n-1)$ edges. A stronger version of the Erdős–Gallai Theorem was given by Kopylov: If G is a 2-connected n-vertex graph with no cycle of length at least k, then $e(G) \leq \max\{h(n,k,2), h(n,k,\lfloor\frac{k-1}{2}\rfloor)\}$, where $h(n,k,a) := \binom{k-a}{2} + a(n-k+a)$. Furthermore, Kopylov presented the two possible extremal graphs, one with $h(n,k,2)$ edges and one with $h(n,k,\lfloor\frac{k-1}{2}\rfloor)$ edges.

In this paper, we complete a stability theorem which strengthens Kopylov's result. In particular, we show that for $k \geq 3$ odd and all $n \geq k$, every n-vertex 2-connected graph G with no cycle of length at least k is a subgraph of one of the two extremal graphs or $e(G) \leq \max\{h(n,k,3), h(n,k,\frac{k-3}{2})\}$. The upper bound for $e(G)$ here is tight.

1. Introduction

One of the fundamental questions in extremal graph theory is to determine the maximum number of edges in an n-vertex graph with no k-vertex path. According to [8], this problem was posed by Turán. A solution to the problem was obtained by Erdős and Gallai [4]:

Theorem 1.1 (*Erdős and Gallai [4]*). *Let G be an n-vertex graph with more than $\frac{1}{2}(k-2)n$ edges, $k \geq 2$. Then G contains a k-vertex path P_k.*

Theorem 1.1 can be proved as a corollary of the following theorem about cycles in graphs:

Theorem 1.2 (*Erdős and Gallai [4]*). *Fix $n, k \geq 3$. If G is an n-vertex graph that does not contain a cycle of length at least k, then $e(G) \leq \frac{1}{2}(k-1)(n-1)$.*

The bound of Theorem 1.2 is best possible for $n-1$ divisible by $k-2$. Indeed, any connected n-vertex graph in which every block is a K_{k-1} has $\frac{1}{2}(k-1)(n-1)$ edges and no cycles of length at least k. In the 1970s, some refinements and new proofs of Theorem 1.2 were obtained by Faudree and Schelp [6,5], Lewin [10], and Woodall [11]—see [8] for more details. The strongest version was proved by Kopylov [9]. His result uses the following n-vertex graphs $H_{n,k,a}$, where $n \geq k$ and $1 \leq a < \frac{1}{2}k$. The vertex set of $H_{n,k,a}$ is the union of three disjoint sets A, B, and C such that $|A| = a$, $|B| = n-k+a$ and

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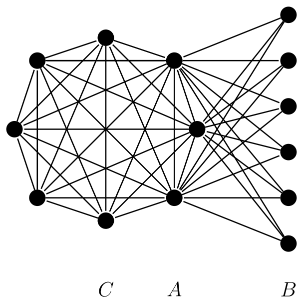

Fig. 1. $H_{14,11,3}$.

$|C| = k - 2a$, and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$ (Fig. 1 shows $H_{14,11,3}$). Let

$$h(n, k, a) := e(H_{n,k,a}) = \binom{k-a}{2} + a(n - k + a).$$

For a graph G containing a cycle, the *circumference*, $c(G)$, is the length of a longest cycle in G. Observe that $c(H_{n,k,a}) < k$: Since $|A \cup C| = k - a$, any cycle D of at length at least k has at least a vertices in B. But as B is independent and $2a < k$, D also has to contain at least $k + 1$ neighbors of the vertices in B, while only a vertices in A have neighbors in A. Kopylov [9] showed that the extremal 2-connected n-vertex graphs with no cycles of length at least k are $G = H_{n,k,2}$ and $G = H_{n,k,t}$ (see Fig. 2): the first has more edges for small n, and the second has more edges for large n.

Theorem 1.3 (*Kopylov [9]*). *Let $n \geq k \geq 5$ and $t = \lfloor \frac{1}{2}(k-1) \rfloor$. If G is an n-vertex 2-connected graph with $c(G) < k$, then*

$$e(G) \leq \max\{h(n, k, 2), h(n, k, t)\} \tag{1}$$

with equality only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

Kopylov's theorem also implies Theorem 1.2 by applying induction to each block of a graph.

2. Results

2.1. A previous result

Recently, three of the present authors proved in [7] a stability version of Theorems 1.2 and 1.3 for n-vertex 2-connected graphs with $n \geq 3k/2$, but the problem remained open for $n < 3k/2$ when $k \geq 9$. The main result of [7] was the following:

Theorem 2.1 (*Füredi, Kostochka, Verstraëte [7]*). *Let $t \geq 2$ and $n \geq 3t$ and $k \in \{2t + 1, 2t + 2\}$. Let G be a 2-connected n-vertex graph $c(G) < k$. Then $e(G) \leq h(n, k, t - 1)$ unless*

(*a*) *$k = 2t + 1$, $k \neq 7$, and $G \subseteq H_{n,k,t}$ or*
(*b*) *$k = 2t + 2$ or $k = 7$, and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t.*

2.2. The essence of the main result

The paper [7] also describes the 2-connected n-vertex graphs G with $e(G) > h(n, k, t - 1)$ and $c(G) < k \leq 8$ for all $n \geq k$. In particular, for $k < 8$, each such graph satisfies either (*a*) or (*b*) of Theorem 2.1.

Together with the cases for $k \leq 8$, this paper gives a full description of the 2-connected n-vertex graphs G with $c(G) < k$ and 'many' edges for all k and n. Our main result is:

Theorem 2.2. *Let $t \geq 4$ and $k \in \{2t + 1, 2t + 2\}$, so that $k \geq 9$. If G is a 2-connected graph on $n \geq k$ vertices and $c(G) < k$, then either $e(G) \leq \max\{h(n, k, t - 1), h(n, k, 3)\}$ or*

(*a*) *$k = 2t + 1$ and $G \subseteq H_{n,k,t}$ or*
(*b*) *$k = 2t + 2$ and $G - A$ is a star forest for some $A \subseteq V(G)$ of size at most t.*
(*c*) *$G \subseteq H_{n,k,2}$.*

Note that

$$h(n, k, t) - h(n, k, t - 1) = \begin{cases} n - t - 3 & \text{if } k = 2t + 1, \\ n - t - 5 & \text{if } k = 2t + 2, \end{cases}$$