Discrete Mathematics 341 (2018) 1749–1754

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

New injective proofs of the Erdős–Ko–Rado and Hilton–Milner theorems

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ARTICLE INFO

Article history:
Received 16 February 2017
Received in revised form 11 March 2018
Accepted 12 March 2018
Available online 1 April 2018

Keywords:
05D05
Intersecting families
Erdős–Ko–Rado
Hilton–Milner
Shifting technique
Injective proof

ABSTRACT

A set system $\mathcal{F}$ is *intersecting* if for any $F, F' \in \mathcal{F}$, $F \cap F' \neq \emptyset$. A fundamental theorem of Erdős, Ko and Rado states that if $\mathcal{F}$ is an intersecting family of r-subsets of $[n] = \{1, \ldots, n\}$, and $n \geq 2r$, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. Furthermore, when $n > 2r$, equality holds if and only if $\mathcal{F}$ is the family of all r-subsets of $[n]$ containing a fixed element. This was proved as part of a stronger result by Hilton and Milner. In this note, we provide new injective proofs of the Erdős–Ko–Rado and the Hilton–Milner theorems.

1. The Erdős–Ko–Rado theorem

For $0 \leq j \leq n$, let $[j, n] = \{j, \ldots, n\}$. In particular, set $[n] = [1, n]$. Similarly, define $(j, n) = \{j+1, \ldots, n-1\}$. For a set X and $1 \leq r \leq |X|$, denote $2^X = \{A : A \subseteq X\}$ and $\binom{X}{r} = \{A \in 2^X : |A| = r\}$. A family $\mathcal{F} \subseteq \binom{[n]}{r}$ is called *r-uniform*, with $\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$ called its *star centered at x*. A *full star* is $\binom{[n]}{r}_x$ for some x; it is easy to see that $|\binom{[n]}{r}_x| = \binom{n-1}{r-1}$. We say that $\mathcal{F}$ is *intersecting* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$.

One of the central results in extremal set theory, the Erdős–Ko–Rado theorem finds a tight upper bound on the size of uniform intersecting set systems. As part of a stronger result that characterized the size and structure of the "second best" intersecting set systems, Hilton and Milner [14] proved that the extremal structures are essentially (up to isomorphism) unique.

Theorem 1 ([7,14]). *If $1 \leq r \leq n/2$ and $\mathcal{F} \subseteq \binom{[n]}{r}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. If $r < n/2$ then equality holds if and only if $\mathcal{F} = \binom{[n]}{r}_x$ for some $x \in [n]$.*

A cornerstone of extremal combinatorics, the theorem has inspired a multitude of research avenues and applications (see [6,8,12,13,15]). The original proof by Erdős, Ko and Rado made use of the now-central *shifting* technique in conjunction with an induction argument. Daykin [5] later discovered that the theorem is implied by the Kruskal–Katona theorem [17,20], while Katona [18] gave possibly the simplest proof using the notion of *cyclic permutations*. Most recently, Frankl and Füredi [10] provided another new short proof of the theorem using a non-trivial result of Katona [16] on shadows of intersecting families.

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https://doi.org/10.1016/j.disc.2018.03.010

The new proof we provide is closest in spirit to the original proof, but avoids induction and counting, and is as short as any. It relies on the shifting operation and some of its structural properties to construct an injective function that maps any intersecting family to a subfamily of $\binom{[n]}{r}_1$. While the shifting operation is injective, it is not explicitly so; that is, the shift operation on a set depends on the entire family. However, our new injection for shifted families is explicit. By direct comparison, while the approach of [10] uses an explicit complementation followed by a shadow bound, our approach uses shifting followed by an explicit complementation. Finally, as mentioned earlier, our technique also helps recover a new short proof of the Hilton–Milner theorem (Theorem 11), which we describe in the final section. We also note here that Borg [4] used an injective argument to prove an analog of the Erdős–Ko–Rado theorem for integer partitions.

2. Shifting

We begin by reviewing the definition of the renowned shifting operation and state some of its important properties. For set $A \subseteq [n]$ and $x \in [n]$, let $A + x = A \cup \{x\}$, $A - x = A \setminus \{x\}$.

Define the (i, j)-*shift* $\sigma_{i,j} : 2^{[n]} \to 2^{[n]}$ as follows: for $A \in 2^{[n]}$, let $\sigma_{i,j}(A) = A - i + j$ if $i \in A$ and $j \notin A$, and $\sigma_{i,j}(A) = A$ otherwise. Extend this definition to $\sigma_{i,j} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ as follows: for $\mathcal{F} \subseteq 2^{[n]}$, let $\sigma_{i,j}(\mathcal{F}) = \{\sigma'_{i,j}(A) : A \in \mathcal{F}\}$, where $\sigma'_{i,j}(A) = \sigma_{i,j}(A)$ if $\sigma_{i,j}(A) \notin \mathcal{F}$, and $\sigma'_{i,j}(A) = A$ otherwise. The following facts are well known and easy to verify.

Fact 2. *For all $A \subseteq [n]$ and all $\mathcal{F} \subseteq 2^{[n]}$ we have*

1. $|\sigma_{i,j}(A)| = |A|$,
2. $|\sigma_{i,j}(\mathcal{F})| = |\mathcal{F}|$, *and*
3. *If $\mathcal{F}$ is intersecting then so is $\sigma_{i,j}(\mathcal{F})$.*

We say that a family $\mathcal{F} \subseteq \binom{[n]}{r}$ is *shifted* if for any $1 \le j < i \le n$, $\sigma_{i,j}(\mathcal{F}) = \mathcal{F}$. Frankl [8] proved the following useful proposition about shifted families.

Proposition 3. *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be shifted and intersecting. Then for every $F \in \mathcal{F}$, there exists a $k = k(F)$ such that $|F \cap [2k+1]| \ge k + 1$.*

The following corollary of Proposition 3 is immediate, and will be used in the proof of Claim 5.

Corollary 4. *Let $\mathcal{F} \subseteq \binom{[n]}{r}$ be shifted and intersecting, and let $r \le n/2$. Then for every $F \in \mathcal{F}$, there exists a $k = k(F)$ such that $|F \cap [2k]| = k$.*

Proof. Let $F \in \mathcal{F}$ and let $k = k(F)$ be maximum such that $|F \cap [2k]| \ge k$. From Proposition 3, we know that such a k exists. We claim that $|F \cap [2k]| = k$. If $2k = n$, then we have $|F \cap [n]| = r \le \frac{1}{2}(2k) = k$, which implies the result, so we assume that $2k < n$. Suppose that $|F \cap [2k]| \ge k + 1$. First, this implies that $n \ge 2k + 2$. Next, the maximality of k implies that $|F \cap [2k+2]| \le k$, a contradiction. Thus $|F \cap [2k]| = k$. □

3. Proof of Theorem 1

For intersecting $\mathcal{F} \subseteq \binom{[n]}{r}$ with $1 \le r \le n/2$, we shift $\mathcal{F}$ until it becomes the shifted, intersecting family $\mathcal{F}'$. Now define the function $\phi : \mathcal{F}' \to \binom{[n]}{r}_1$ as follows. For a set $F \in \mathcal{F}'$, let $\kappa = \kappa_F$ be maximum such that $|F \cap [2\kappa]| = \kappa$. We know that κ exists, from Corollary 4. Now, if $1 \in F$, let $\phi(F) = F$; otherwise, let $\phi(F) = F \triangle [2\kappa]$. We also denote $\phi(\mathcal{F}) = \{\phi(A) : A \in \mathcal{F}\}$, as well as write $\phi^{-1}(B) = A$ whenever $\phi(A) = B$, with $\phi^{-1}(\mathcal{H}) = \{\phi^{-1}(B) : B \in \mathcal{H}\}$.

Fact 2 gives $|\mathcal{F}| = |\mathcal{F}'|$, and Claim 5 below gives $|\mathcal{F}'| \le \binom{n-1}{r-1}$. When $r < n/2$, Lemma 10 shows that $\mathcal{F}'$ is a full star, and Lemma 9 below shows that $\mathcal{F}$ is a full star. □

We now prove Claim 5 and Lemmas 9 and 10 in the subsections below.

3.1. Injection

Claim 5. *For $r \le n/2$, if $\mathcal{F} \subseteq \binom{[n]}{r}$ is shifted and intersecting then the function ϕ defined above is injective.*

Proof. Let $F_1, F_2 \in \mathcal{F}$, $F_1 \ne F_2$. If $1 \in F_1$ and $1 \in F_2$ then it is obvious that $\phi(F_1) \ne \phi(F_2)$.

Suppose that $1 \notin F_1$ and $1 \notin F_2$. If $\kappa = \kappa_{F_1} = \kappa_{F_2}$ then $F_1 \cap [2\kappa] \ne F_2 \cap [2\kappa]$ or $F_1 \setminus [2\kappa] \ne F_2 \setminus [2\kappa]$. Then the definition of ϕ implies that $\phi(F_1) \ne \phi(F_2)$, as required. So, without loss of generality, we may assume that $\kappa_{F_1} < \kappa_{F_2}$. Using maximality of κ_{F_1}, we have that $F_1 \setminus [2\kappa_{F_2}] \ne F_2 \setminus [2\kappa_{F_2}]$. As $F_1 \setminus [2\kappa_{F_2}] \subseteq \phi(F_1)$ and $F_2 \setminus [2\kappa_{F_2}] \subseteq \phi(F_2)$, this implies that $\phi(F_1) \ne \phi(F_2)$.

Finally, suppose $1 \in F_1$ and $1 \notin F_2$. We need to show that $\phi(F_2) \ne F_1$. Suppose instead that $\phi(F_2) = F_1$. Suppose $\kappa_{F_2} = r$. Then $\phi(F_2) = [2r] \setminus F_2$. As $\phi(F_2) = F_1$, $F_1 \cap F_2 = \emptyset$, a contradiction. Thus we may assume $\kappa_{F_2} < r$. Let $G_2 = F_2 \setminus [2\kappa_{F_2}]$ (note that $|G_2| > 0$ as $\kappa_{F_2} < r$) and break G_2 into its maximum intervals. That is, we write $G_2 = \cup_{i=0}^{p-1}[t_i, s_{i+1}]$, where $2\kappa_{F_2} = s_0 < t_0$, $t_i \le s_{i+1}$ for each $0 \le i < p$, $s_i + 1 < t_i$ for each $0 < i < p$, and $t_p = n$. For every $h \in [0, p-1]$, we can see that $|\cup_{i=0}^{h}(s_i, t_i)| > |\cup_{i=0}^{h}[t_i, s_{i+1}]|$, which we refer to as *Property* ⋆. Indeed, for each $1 \le j \le n$ define $X_j = |[j] \setminus F_2| - |F_2 \cap [j]|$.