ARTICLE IN PRESS
Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On graphs with largest possible game domination number

Kexiang Xu [a], Xia Li [a], Sandi Klavžar [b,c,d,*]

[a] College of Science, Nanjing University of Aeronautics & Astronautics, Nanjing, Jiangsu, 210016, PR China
[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

ARTICLE INFO

Article history:
Received 7 April 2017
Received in revised form 19 October 2017
Accepted 23 October 2017
Available online xxxx

Keywords:
Domination game
γ_g-maximal graph
Supportive dominating set
Starlike tree

ABSTRACT

Let $\gamma(G)$ and $\gamma_g(G)$ be the domination number and the game domination number of a graph G, respectively. In this paper γ_g-maximal graphs are introduced as the graphs G for which $\gamma_g(G) = 2\gamma(G) - 1$ holds. Large families of γ_g-maximal graphs are constructed among the graphs in which their sets of support vertices are minimum dominating sets. γ_g-maximal graphs are also characterized among the starlike trees, that is, trees which have exactly one vertex of degree at least 3.

1. Introduction

If $G = (V(G), E(G))$ is a graph, then a vertex $u \in V(G)$ *dominates* a vertex $v \in V(G)$ if $u = v$ or u is adjacent to v. $S \subseteq V(G)$ is a *dominating set* of G if every vertex in G is dominated by a vertex in S. The size of a smallest dominating set of G is the *domination number* $\gamma(G)$ of G. A smallest dominating set will be briefly called a γ-*set*.

The domination game is played on a graph G by two players that are usually called *Dominator* and *Staller*. They take turns choosing a vertex from G such that at least one previously undominated vertex becomes dominated until no move is possible. The score of the game is the total number of vertices chosen by them in this game. The players have opposite goals: Dominator wants to minimize the score and Staller wants to maximize it. A game is called a *D-game* (resp. *S-game*) if Dominator (resp. Staller) has the first move. The *game domination number* $\gamma_g(G)$ of G is the score of a D-game played on G assuming that both players play optimally, the *Staller-start game domination number* $\gamma_g'(G)$ is the score of an optimal S-game.

This game was introduced in [4] and investigated by now in about 30 papers. One of the reasons for this large interest is the 3/5-conjecture due to Kinnersley, West and Zamani asserting that $\gamma_g(G) \leq 3|V(G)|/5$ holds for any isolate-free graph G [17, Conjecture 6.2]. (Related conjectures were stated also for the S-game, as well as for both games played on forests.) Bujtás [6,7] developed an innovative discharging-like method to attack this conjecture. Using the method, the conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [11]. Along these lines Schmidt [23] determined a largest known class of trees for which the conjecture holds. Moreover, Marcus and Peleg reported in arXiv [21] that the conjecture holds on all isolate-free forests. Among the other aspects of the domination game we list here: domination game critical graphs [8]; the somehow peculiar behavior of the game on unions of graphs [10]; graphs with small game domination number [18]; different realizations of the game domination number [19]; a characterization of forests with the game domination number equal to the domination number [22]; bluffing aspects of the domination game [1]; and the PSPACE-completeness of the game domination number [2]. We also mention two related games that were

* Corresponding author at: Faculty of Mathematics and Physics, University of Ljubljana, Slovenia.
E-mail addresses: kexxu1221@126.com (K. Xu), nhlxylixia@163.com (X. Li), sandi.klavzar@fmf.uni-lj.si (S. Klavžar).

https://doi.org/10.1016/j.disc.2017.10.024
0012-365X/

Please cite this article in press as: K. Xu, et al., On graphs with largest possible game domination number, Discrete Mathematics (2017), https://doi.org/10.1016/j.disc.2017.10.024.

introduced based on the domination game: the total domination game [12] (see also [13,15]) and the disjoint domination game [9].

It was shown in [4, Theorem 1] that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ holds for any graph G. Moreover, all possible values for γ_g are eventually realizable [4, Theorem 10]. It is hence natural to ask for which graphs G the equalities $\gamma_g(G) = \gamma(G)$ and $\gamma_g(G) = 2\gamma(G) - 1$ hold, respectively. The former problem was solved for the case of trees in [22], where it was also conjectured that if G is a connected graph with $\gamma_g(G) = \gamma(G)$, then G is either a tree or has girth at most 7. The general problem to characterize the graphs G with $\gamma_g(G) = \gamma(G)$ seems highly difficult though. In this paper we consider the other extreme case, that is, which graphs G have the largest possible game domination number $2\gamma(G) - 1$. We will call such graphs γ_g-*maximal*.

In the next section additional concepts needed are introduced, several known results to be used later recalled, and a couple of useful facts deduced. In Section 3 large families of γ_g-maximal graphs are constructed among the graphs in which their sets of support vertices (vertices adjacent to leaves) are γ-sets. In the last two sections we consider trees which have exactly one vertex of degree at least 3, called starlike trees. In Section 4 we characterize γ_g-maximal starlike trees among the starlike trees with at least one 1-arm, while in Section 5 we characterize γ_g-maximal starlike trees among the other starlike trees. In the concluding section we observe that the graphs considered in this paper support the 3/5-conjecture.

2. Preliminaries

We will use the notation $[k] = \{1, \ldots, k\}$ for a positive integer k. The maximum degree and the minimum degree in a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex v of G with $\deg_G(v) = 1$ is called a *pendant vertex* (alias *leaf*), the vertex adjacent to v is a *support vertex* (to v). Let $L(G)$ and $\mathrm{Supp}(G)$ denote the set of pendant and support vertices of G, respectively. For a vertex v of G let $L(v) = L(G) \cap N(v)$, where $N(v)$ is the open neighborhood of v. Clearly, $L(v) \neq \emptyset$ if and only if $v \in \mathrm{Supp}(G)$. Note also that $L(K_2) = \mathrm{Supp}(K_2) = V(K_2)$. On the other hand, if G is connected and of order at least 3, then a support vertex is of degree at least 2, so that $L(G) \cap \mathrm{Supp}(G) = \emptyset$.

Suppose that a D-game is played. Then we will denote the sequence of vertices selected by Dominator with $d_1, d_2, \ldots,$ and with $s_1, s_2, \ldots$ the sequence chosen by Staller. A *partially-dominated graph* is a graph G together with a declaration that some vertices $S \subseteq V(G)$ are already dominated in the sense that they need not be dominated in the rest of the game. It is denoted with $G|S$.

We next recall the following fundamental results to be used later.

Lemma 2.1 (*Continuation Principle, [17]*). *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma_g'(G|A) \leq \gamma_g'(G|B)$.*

Theorem 2.2 (*[17]*). *If F is a forest with $S \subseteq V(F)$, then $\gamma_g(F|S) \leq \gamma_g'(F|S)$.*

Theorem 2.3 (*[4,17]*). *If G is any graph, then $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.*

Setting $S = \emptyset$ in Theorem 2.2 and specializing to trees we get:

Corollary 2.4. *If T is a tree, then $\gamma_g(T) \leq \gamma_g'(T)$.*

Denoting with $G \cup H$ the disjoint union of graphs G and H we have the following result that will be useful to us.

Lemma 2.5 (*[17, Lemma 5.4]*). *If F_1 and F_2 are partially dominated forests, then*

$$\gamma_g(F_1 \cup F_2) \leq \gamma_g(F_1) + \gamma_g'(F_2) \quad \text{and} \quad \gamma_g'(F_1 \cup F_2) \leq \gamma_g'(F_1) + \gamma_g'(F_2).$$

The next result was first proved in the unpublished manuscript [16]. Five years later the first published proof appeared in [20].

Theorem 2.6 (*[16,20]*). *If $n \geq 1$, then*

(i) $\gamma_g(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil; & \textit{otherwise}. \end{cases}$

(ii) $\gamma_g'(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

Following the notation from [20], let P_n' denote the partially dominated path of order $n+1$ with one of its leaves dominated. Then from Košmrlj's proof of [20, Theorem 2.6] we extract the following information useful to us.

Lemma 2.7. *If S-game is played on P_n' or a union of some P_{n_i}'s, then some dominated leaf is an optimal move for Staller. Moreover, if $n \geq 1$, then*
(i) $\gamma_g(P_n) = \gamma_g(P_n')$, and
(ii) $\gamma_g(P_{n+3}) = 1 + \gamma_g'(P_n')$.