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An infinite family of cubic nonnormal Cayley graphs on nonabelian simple groups

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ABSTRACT

We construct a connected cubic nonnormal Cayley graph on A_{2^m-1} for each integer $m \geqslant 4$ and determine its full automorphism group. This is the first infinite family of connected cubic nonnormal Cayley graphs on nonabelian simple groups.

1. Introduction

In this paper all graphs considered are finite, simple and undirected. Given a group G and an inverse-closed subset S of $G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is the graph with vertex set G such that two vertices x and y are adjacent if and only if $yx^{-1} \in S$. Let $\widehat{G}$ be the right regular representation of G. It is easy to see that $\widehat{G}$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Moreover, it was shown by Godsil [5] that the normalizer of $\widehat{G}$ in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ is $\widehat{G} \rtimes \mathrm{Aut}(G, S)$, where $\mathrm{Aut}(G, S)$ is the group of automorphisms of G fixing S setwise. In particular, $\mathrm{Aut}(\mathrm{Cay}(G, S)) = \widehat{G} \rtimes \mathrm{Aut}(G, S)$ if and only if $\widehat{G}$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Viewing this, Xu in [14] introduced the concept of normal Cayley graphs: a Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\widehat{G}$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. The study of normality of a Cayley graph plays an important role in the study of its automorphism group because once a Cayley graph $\mathrm{Cay}(G, S)$ is known to be normal, to determine its full automorphism group one only needs to determine the group $\mathrm{Aut}(G, S)$, which is usually much easier. For a survey paper on normality of Cayley graphs we refer the reader to [4].

The normality of cubic Cayley graphs on nonabelian simple groups has received considerable attention. It was proved in [12] that a connected cubic Cayley graph $\mathrm{Cay}(G, S)$ with G nonabelian simple is normal if $\widehat{G} \rtimes \mathrm{Aut}(G, S)$ is transitive on the edge set of $\mathrm{Cay}(G, S)$. A graph is said to be *arc-transitive* if its automorphism group acts transitively on the set of arcs. In [15,16] it was proved that the only connected arc-transitive cubic nonnormal Cayley graphs on nonabelian simple groups are two Cayley graphs on A_{47} up to isomorphism, and their full automorphism groups are both isomorphic to A_{48}. On the other hand, examples of connected cubic nonnormal Cayley graphs on nonabelian simple groups are very rare. Since the connected arc-transitive cubic nonnormal Cayley graphs on nonabelian simple groups are only the above mentioned two graphs on A_{47}, we can concentrate on the non-arc-transitive case. In this context, one has the following theorem combining [2, Theorem 1.1] and [17, Theorem 1.2].

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Theorem 1.1 ([2,17]). *Let* Cay(G, S) *be a connected cubic nonnormal Cayley graph on a nonabelian simple group* G. *If* Cay(G, S) *is not arc-transitive, then one of the following holds:*

(a) $G = \mathrm{A}_{2^m-1}$ *with* $m \geqslant 3$;
(b) G *is a simple group of Lie type of even characteristic except* $\mathrm{PSL}_2(2^e)$, $\mathrm{PSL}_3(2^e)$, $\mathrm{PSU}_3(2^e)$, $\mathrm{PSp}_4(2^e)$, $\mathrm{E}_8(2^e)$, $\mathrm{F}_4(2^e)$, ${}^2\mathrm{F}_4(2^e)'$, $\mathrm{G}_2(2^e)$ *and* $\mathrm{Sz}(2^e)$.

Until recently, connected cubic nonnormal Cayley graphs on the groups listed in Theorem 1.1 were only found for A_{15} and A_{31} [9]. In 2008, Feng, Lu and Xu asked the following question in their survey paper [4] on normality of Cayley graphs.

Question 1.2 ([4, Problem 5.9]). *Are there infinitely many connected nonnormal Cayley graphs of valency* 3 *or* 4 *on nonabelian simple groups?*

Question 1.2 in the valency 4 case has been answered by Wang and Feng [13] in the affirmative. In this paper, we answer the question in the remaining case. Our main result is Theorem 1.3, which gives a positive answer to Question 1.2.

Theorem 1.3. *For each integer* $m \geqslant 4$, *there exists a graph* Γ_m *satisfying:*

(a) Γ_m *is a connected cubic nonnormal Cayley graph on* A_{2^m-1};
(b) $\Gamma_m \cong \mathrm{Cay}(\mathrm{A}_{2^m-1}, S)$ *for some set* S *of three involutions in* A_{2^m-1} *such that* $\mathrm{Aut}(\mathrm{A}_{2^m-1}, S) = 1$;
(c) $\mathrm{Aut}(\Gamma_m) \cong \mathrm{A}_{2^m}$.

We call a Cayley graph Cay(G, S) a *graphical regular representation* (*GRR* for short) of G if $\mathrm{Aut}(\mathrm{Cay}(G, S)) = \widehat{G}$. Note that a GRR is necessarily a normal Cayley graph, and a necessary condition for Cay(G, S) to be a GRR is that $\mathrm{Aut}(G, S) = 1$. In many circumstances it is shown that this condition is also sufficient, see for example [2,5,6]. More generally, a problem is posed in [2] to determine the groups G such that a Cayley graph Cay(G, S) on G is a GRR of G if and only if $\mathrm{Aut}(G, S) = 1$. We remark that our graph Γ_m in Theorem 1.3 as a Cayley graph on $G := \mathrm{A}_{2^m-1}$ is not only nonnormal (and hence not a GRR) but also satisfies the condition $\mathrm{Aut}(G, S) = 1$. It is also worth remarking that, although the graph Γ_m is not arc-transitive, it has local action C_2 so that it corresponds to a tetravalent arc-transitive graph in the standard way described in [11, Section 4.1].

The paper is organized as follows. We shall first give the construction of Γ_m for Theorem 1.3 in Section 2. Then the entirety of Section 3 will be devoted to proving the connectivity of Γ_m. Finally in Section 4 we prove the remaining properties of Γ_m described in Theorem 1.3, thus completing the proof of the theorem.

2. Construction of Γ_m

We first introduce some notation that is fixed throughout this paper. Let $m \geqslant 4$ be an integer,

$$H = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1\rangle \times \langle c_1\rangle \times \langle c_2\rangle \times \cdots \times \langle c_{m-3}\rangle,$$

where $c_1, c_2, \ldots, c_{m-3}$ are involutions,

$$K = \langle a^2, b, c_1, c_2, \ldots, c_{m-3}\rangle = \langle a^2\rangle \times \langle b\rangle \times \langle c_1\rangle \times \langle c_2\rangle \times \cdots \times \langle c_{m-3}\rangle$$

and $h = a\prod_{i=0}^{\lceil (m-5)/2\rceil} c_{2i+1}$. Clearly, H is the direct product of a dihedral group D_8 of order 8 and an elementary abelian 2-group of rank $m-3$, so that $|H| = 2^m$. For the sake of convenience, put $c_{-1} = c_0 = 1$. Define $x \in \mathrm{Aut}(H)$ by letting

$$a^x = a^{-1}, \quad b^x = ab, \quad c_{2i+1}^x = c_{2i+1} \quad \text{and} \quad c_{2i+2}^x = a^2 c_{2i+1}c_{2i+2}$$

for $0 \leqslant i \leqslant \lfloor (m-5)/2\rfloor$ and letting $c_{m-3}^x = a^2 c_{m-3}$ in addition if m is even. Define $\tau \in \mathrm{Aut}(K)$ by letting

$$(a^2)^\tau = b, \quad b^\tau = a^2, \quad c_{2i+1}^\tau = c_{2i-1}c_{2i}c_{2i+2} \quad \text{and} \quad c_{2i+2}^\tau = c_{2i-1}c_{2i}c_{2i+1}$$

for $0 \leqslant i \leqslant \lfloor (m-5)/2\rfloor$ and letting $c_{m-3}^\tau = c_{m-3}$ in addition if m is even. Note that x and τ are indeed automorphisms of H and K because the images of generators under x and τ satisfy the defining relation for H and K, respectively. Denote the right regular representation of H by R. Let y be the permutation of H such that $g^y = g^\tau$ and

$$(hg)^y = \begin{cases} hg^\tau & \text{if } m \text{ is odd,} \\ hg^\tau c_{m-3} & \text{if } m \text{ is even} \end{cases}$$

for $g \in K$. Let

$$z = \begin{cases} R(h)yR(h^{-1}) & \text{if } m \text{ is odd,} \\ R(h)yR(h^{-1}c_{m-3}) & \text{if } m \text{ is even.} \end{cases}$$

We will see that the three permutations x, y and z of H are all involutions in $\mathrm{Alt}(H)$.

Lemma 2.1. *x, y and z are all involutions.*