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Hamiltonian properties of 3-connected {claw,hourglass}-free graphs

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ABSTRACT

We show that some sufficient conditions for hamiltonian properties of claw-free graphs can be substantially strengthened under an additional assumption that G is hourglass-free (where hourglass is the graph with degree sequence 4, 2, 2, 2, 2).

Let G be a 3-connected claw-free and hourglass-free graph of order n. We show that

(i) if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i+j+k \leq 9$, then G is hamiltonian,
(ii) if G is P_{12}-free, then G is Hamilton-connected,
(iii) G contains a cycle of length at least $\min\{\sigma_{12}(G), n\}$, unless $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph,
(iv) if $\sigma_{13}(G) \geq n+1$, then G is hamiltonian, unless $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Here P_i denotes the path on i vertices, Z_i ($N_{i,j,k}$) denotes the graph obtained by attaching a path of length $i \geq 1$ (three vertex-disjoint paths of lengths $i, j, k \geq 1$) to a triangle, $\sigma_k(G)$ denotes the minimum degree sum over all independent sets of size k, and $L^{-1}(\mathrm{cl}(G))$ is the line graph preimage of the closure of G.

1. Introduction

We basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [2].

Specifically, by a *graph* we always mean a simple finite graph $G = (V(G), E(G))$; in some situations, where we admit multiple edges (specifically, in Section 3.3), we speak about a *multigraph*.

We use $N_G(x)$ to denote the neighborhood and $d_G(x)$ to denote the degree of a vertex $x \in V(G)$. A *pendant vertex* is a vertex of degree 1, and a *pendant edge* is an edge having a pendant vertex. We denote $V_i(G) = \{x \in V(G)|\ d_G(x) = i\}$, $V_{\leq i}(G) = \{x \in V(G)|\ d_G(x) \leq i\}$, and $V_{\geq i}(G) = \{x \in V(G)|\ d_G(x) \geq i\}$. We use $\delta(G)$ to denote the minimum degree of G, and, for a positive integer k, we set $\sigma_k(G) = \min\{\sum_{x\in I} d_G(x)|\ I \subset V(G)$ independent, $|I| = k\}$ if G contains an independent set of size k, and $\sigma_k(G) = \infty$ otherwise. For $M \subset V(G)$, $\langle M\rangle_G$ denotes the induced subgraph on M. If F, G are graphs, we write $F \subset G$ if F is a subgraph of G, $F \overset{\mathrm{IND}}{\subset} G$ if F is an induced subgraph of G, and $F \simeq G$ if F and G are isomorphic. By a *clique* we mean a complete subgraph of G, not necessarily maximal, and we say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique.

For a set $X \subset E(G)$, an *X-contraction of G* is the graph $G|_X$ obtained from G by identifying the vertices of each edge in X and removing the resulting loops. For a connected subgraph $F \subset G$, we set $G|_F = G|_{E(F)}$, we use $\mathrm{con}(F)$ to denote the vertex in $G|_F$

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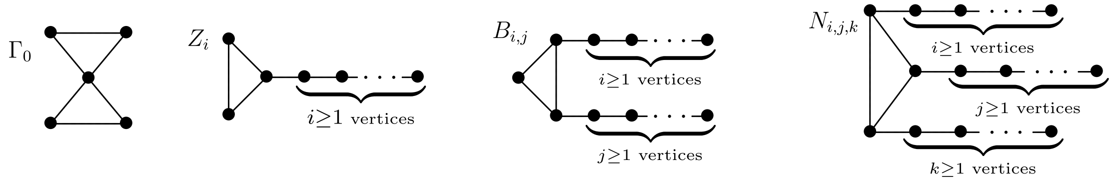

Fig. 1. The graphs Γ_0, Z_i, $B_{i,j}$ and $N_{i,j,k}$.

to which F is contracted, and we also say that F is the *contraction preimage* of the vertex $v = \mathrm{con}(F)$, denoted $F = \mathrm{con}^{-1}(v)$. Finally, if F and G are graphs, we say that G *has a nontrivial contraction to* F if there is $X \subset E(G)$ such that $G|_X \simeq F$ and for every $v \in V(F)$, $\mathrm{con}^{-1}(v)$ is nontrivial.

Throughout the paper, $c(G)$ denotes the *circumference of* G, i.e., the length of a longest cycle in G. A graph G is *hamiltonian* if $c(G) = |V(G)|$, i.e., if G contains a *hamiltonian cycle*, and G is *Hamilton-connected* if, for any $x, y \in V(G)$, G contains a *hamiltonian* (x, y)*-path*, i.e., an (x, y)-path containing all vertices of G.

If $\mathcal{F}$ is a family of graphs, we say that G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the members of $\mathcal{F}$ are in this context referred to as *forbidden induced subgraphs*. Specifically, for $\mathcal{F} = \{K_{1,3}\}$, we say that G is *claw-free*.

Throughout, P_i denotes the path on i vertices. Further graphs often used as forbidden induced subgraphs are shown in Fig. 1; here the graph Γ_0 is called the *hourglass*, $B_{i,j}$ the *generalized bull* and $N_{i,j,k}$ the *generalized net*.

If H is a graph (multigraph), then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Recall that every line graph is claw-free. It is well-known that if G is a line graph of a graph, then the graph H such that $G = L(H)$ is uniquely determined (with one exception of $G = K_3$). The graph H for which $L(H) = G$ will be called the *preimage* of G and denoted $H = L^{-1}(G)$. We will analogously write $v = L(e)$ and $e = L^{-1}(v)$ for a vertex $v \in V(G)$ and its corresponding edge $e \in E(H)$. However, note that in line graphs of multigraphs this is, in general, not true, as there can be nonisomorphic (multi)graphs with the same line graph. We will discuss this in more detail in Section 3.3, where this will be needed.

A vertex $x \in V(G)$ is said to be *eligible* if $\langle N_G(x)\rangle_G$ is a connected noncomplete graph. We will use $V_{EL}(G)$ to denote the set of all eligible vertices of G. For $x \in V(G)$, the *local completion of* G *at* x is the graph $G^*_x = (V(G), E(G) \cup \{uv|\ u, v \in N_G(x)\})$ (i.e., G^*_x is obtained from G by adding to $\langle N_G(x)\rangle_G$ all missing edges). The *closure* of a claw-free graph G is the graph $\mathrm{cl}(G)$ obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)^*_x$ for some vertex $x \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$, and $G_k = \mathrm{cl}(G)$). We say that G is *closed* if $G = \mathrm{cl}(G)$. The following result summarizes basic properties of the closure operation.

Theorem A (*[18]*). *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}(G)$ *is uniquely determined,*
(ii) $c(\mathrm{cl}(G)) = c(G)$,
(iii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Thus, the closure operation turns a claw-free graph G into a unique line graph of a triangle-free graph while preserving the length of a longest cycle (and hence also the hamiltonicity or nonhamiltonicity) of G.

There are many results on hamiltonian properties of graphs in classes defined in terms of forbidden induced subgraphs. In this paper, we will consider these questions in 3-connected graphs. We first summarize some known results.

Theorem B. *Let G be a 3-connected claw-free graph.*

(i) [17] *If G is P_{11}-free, then G is hamiltonian.*
(ii) [13] *If G is Z_8-free, then G is hamiltonian.*
(iii) [7] *If G is Z_9-free, then either G is hamiltonian, or G is isomorphic to the line graph of the graph obtained from the Petersen graph by adding one pendant edge to each vertex.*
(iv) [10,22] *If G is $N_{i,j,k}$-free with $i + j + k \leq 9$, then G is hamiltonian.*

Note that [22] announces an analogous result for 3-connected $\{K_{1,3}, B_{i,j}\}$-free graphs with $i + j \leq 9$ (with a family of exceptions), however, the proof in [22] is based on the statement that if a graph G is $\{K_{1,3}, B_{i,j}\}$-free, then so is $\mathrm{cl}(G)$, which is known not to be true. Since this is true for $\{K_{1,3}, N_{i,j,k}\}$-free graphs, the proof of (iv) in [22] can be trusted. (Moreover, note that the statement for $\{K_{1,3}, B_{i,j}\}$-free graphs with $i + j \leq 8$ is a direct consequence of (iv)).

Theorem C (*[1]*). *Let G be a 3-connected $\{K_{1,3}, P_9\}$-free graph. Then G is Hamilton-connected.*

There are also many results on degree conditions for hamiltonian properties. We list here the best known ones in 3-connected claw-free graphs.

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