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Local and union boxicity

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ABSTRACT

The boxicity $\operatorname{box}(H)$ of a graph H is the smallest integer d such that H is the intersection of d interval graphs, or equivalently, that H is the intersection graph of axis-aligned boxes in $\mathbb{R}^d$. These intersection representations can be interpreted as covering representations of the complement H^c of H with co-interval graphs, that is, complements of interval graphs. We follow the recent framework of global, local and folded covering numbers (Knauer and Ueckerdt, 2016) to define two new parameters: the local boxicity $\operatorname{box}_\ell(H)$ and the union boxicity $\overline{\operatorname{box}}(H)$ of H. The union boxicity of H is the smallest d such that H^c can be covered with d vertex–disjoint unions of co-interval graphs, while the local boxicity of H is the smallest d such that H^c can be covered with co-interval graphs, at most d at every vertex.

We show that for every graph H we have $\operatorname{box}_\ell(H) \leq \overline{\operatorname{box}}(H) \leq \operatorname{box}(H)$ and that each of these inequalities can be arbitrarily far apart. Moreover, we show that local and union boxicity are also characterized by intersection representations of appropriate axis-aligned boxes in $\mathbb{R}^d$. We demonstrate with a few striking examples, that in a sense, the local boxicity is a better indication for the complexity of a graph, than the classical boxicity.

1. Introduction

All graphs considered in this article are finite, undirected, simple (have neither loops nor multiple edges), and have at least one edge. An *interval graph* is an intersection graph of intervals on the real line.[1] Such a set $\{I(v) \subseteq \mathbb{R} \mid v \in V(H)\}$ of intervals with $vw \in E(H) \Leftrightarrow I(v) \cap I(w) \neq \emptyset$ is called an *interval representation of* H. A box in $\mathbb{R}^d$, also called a *d-dimensional box*, is the Cartesian product of d intervals. The *boxicity* of a graph H, denoted by $\operatorname{box}(H)$, is the least integer d such that H is the intersection graph of d-dimensional boxes, and a corresponding set $\{B(v) \subseteq \mathbb{R}^d \mid v \in V(H)\}$ is a *box representation of* H. The boxicity was introduced by Roberts [17] in 1969 and has many applications in as diverse areas as ecology and operations research [4].

As two d-dimensional boxes intersect if and only if each of the d corresponding pairs of intervals intersect, we have the following more graph theoretic interpretation of the boxicity of a graph; also see Fig. 1(a).

Theorem 1 (*Roberts [17]*). *For a graph H we have $\operatorname{box}(H) \leq d$ if and only if $H = G_1 \cap \cdots \cap G_d$ for some interval graphs $G_1, \ldots, G_d$.*

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[1] Throughout, we shall just say "intervals" and drop the suffix "on the real line". Intervals may be open, half-open, or closed (even though restricting to one kind does not affect the notion of an interval graph) and bounded or unbounded.

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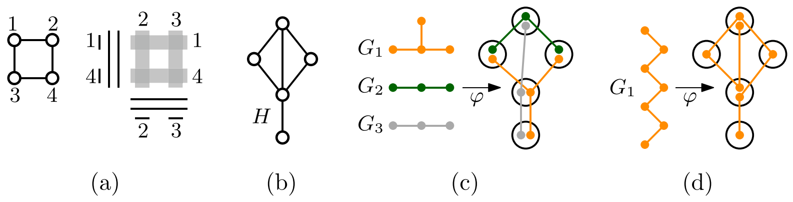

Fig. 1. (a) The 4-cycle as the intersection of two interval graphs. (b) Example graph H. (c) An injective covering of H that is 3-global and 2-local. (d) A (non-injective) 1-global 2-local covering of H.

I.e., the boxicity of a graph H is the least integer d such that H is the intersection of d interval graphs. For a graph $H = (V, E)$ we denote its complement by $H^c = (V, \binom{V}{2} - E)$. Then by De Morgan's law we have

$$H = G_1 \cap \cdots \cap G_d \quad \Longleftrightarrow \quad H^c = G_1^c \cup \cdots \cup G_d^c, \tag{1}$$

i.e., box(H) is the least integer d such that the complement H^c of H is the union of d co-interval graphs $G_1^c, \ldots, G_d^c$, where a *co-interval graph* is the complement of an interval graph.[2] In other words, box(H) $\leq d$ if H^c can be covered with d co-interval graphs. Strictly speaking, we have to be a little more precise here. In order to use De Morgan's law, we should guarantee that $G_1, \ldots, G_d$ in (1) all have the same vertex set. To this end, if G is a subgraph of H, let $\bar{G} = (V(H), E(G))$ be the graph obtained from G by adding all vertices in $V(H) - V(G)$ as isolated vertices. (We use $\bar{G}$ to denote a graph obtained from $G \subseteq H$ by adding vertices of H not in G either as isolated or universal vertices.) Clearly we have

$$H^c = G_1^c \cup \cdots \cup G_d^c \quad \Rightarrow \quad H^c = \bar{G}_1^c \cup \cdots \cup \bar{G}_d^c \quad \Rightarrow \quad H = \bar{G}_1 \cap \cdots \cap \bar{G}_d$$

for any graph H and any set of subgraphs $G_1, \ldots, G_d$ of H. Now whenever G is a co-interval graph, then so is $\bar{G}$, implying that box(H) is the least integer d such that H^c can be covered with d co-interval graphs.

Graph covering parameters

In the general graph covering problem, one is given an input graph H, a so-called covering class $\mathcal{G}$ and a notion of how to cover H with one or more graphs from $\mathcal{G}$. The most classic notion of covering, which also corresponds to the boxicity as discussed above, is that H shall be the union of $G_1, \ldots, G_t \in \mathcal{G}$, i.e., $V(H) = \bigcup_{i\in[t]} V(G_i)$ and $E(H) = \bigcup_{i\in[t]} E(G_i)$. (Here and throughout the paper, for a positive integer t we denote $[t] = \{1, \ldots, t\}$.) The *global covering number*, denoted by $c_g^{\mathcal{G}}(H)$, is then defined to be the minimum t for which such a cover exists. Many important graph parameters can be interpreted as a global covering number, e.g., the arboricity [15] when $\mathcal{G}$ is the class of forests, the track number [9] when $\mathcal{G}$ is the class of interval graphs (Note that this is the smallest number of interval graphs covering a graph H, which is very different from the track-number of H as defined in [5].), and the thickness [1,14] when $\mathcal{G}$ is the class of planar graphs, just to name a few.

Most recently, Knauer and Ueckerdt [10] suggested the following unifying framework for three kinds of covering numbers, differing in the underlying notion of covering. A *graph homomorphism* is a function $\varphi : V(G) \to V(H)$ with the property that if $uv \in E(G)$ then $\varphi(u)\varphi(v) \in E(H)$, i.e., φ maps vertices of G (not necessarily injectively) to vertices of H such that edges are mapped to edges. For abbreviation we shall simply write $\varphi : G \to H$ instead of $\varphi : V(G) \to V(H)$. Whenever G' is a subgraph of G, $\varphi(G')$ denotes the (not necessarily induced) subgraph H' of H with $V(H') = \{\varphi(v) \mid v \in V(G')\}$ and $E(H') = \{\varphi(u)\varphi(v) \mid uv \in E(G')\}$. A *copy* of a graph G' in H is a (not necessarily induced) subgraph H' of H that is isomorphic to G'.

For an input graph H, a covering class $\mathcal{G}$ and a positive integer t, a *t-global $\mathcal{G}$-cover of H* is an edge-surjective homomorphism $\varphi : G_1 \,\dot\cup\, \cdots \,\dot\cup\, G_t \to H$ such that $G_i \in \mathcal{G}$ for each $i \in [t]$. Here $\dot\cup$ denotes the vertex-disjoint union of graphs. We say that φ is *injective* if its restriction to G_i is injective for each $i \in [t]$. A $\mathcal{G}$-cover is called *s-local* if $|\varphi^{-1}(v)| \leq s$ for every $v \in V(H)$.

Hence, if φ is a $\mathcal{G}$-cover of H, then

φ is t-global if it uses only t graphs[3] from the covering class $\mathcal{G}$,
φ is injective if $\varphi(G_i)$ is a copy of G_i in H for each $i \in [t]$,
φ is s-local if for each $v \in V(H)$ at most s vertices are mapped onto v.

[2] Precisely, G is a co-interval graph if there is a set $\{I(v) \subseteq \mathbb{R} \mid v \in V(G)\}$ of intervals with $vw \in E(G) \Leftrightarrow I(v) \cap I(w) = \emptyset$. Equivalently, these are the comparability graphs of interval orders.

[3] More precisely, φ uses a multiset of size t consisting of graphs from $\mathcal{G}$, as the same graph may be used more than once.