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# Altermatic number of categorical product of graphs

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ABSTRACT

In this paper, we prove some relaxations of Hedetniemi's conjecture in terms of altermatic number and strong altermatic number of graphs, two combinatorial parameters introduced by the present authors Alishahi and Hajiabolhassan (2015) providing two sharp lower bounds for the chromatic number of graphs. In terms of these parameters, we also introduce some sharp lower bounds for the chromatic number of the categorical product of two graphs. Using these lower bounds, we present some new families of graphs supporting Hedetniemi's conjecture.



## 1. Introduction

### 1.1. Preliminary notions

Throughout the paper, for two integers $n$ and $k$ with $n \geq k$, the two symbols $[n]$ and $\binom{[n]}{k}$, respectively, stand for the set $\{1, \ldots, n\}$ and the set of all $k$-subsets of $[n]$. Also, unless otherwise stated, we consider only simple graphs (graphs without loops or parallel edges). For two given graphs $G$ and $H$, their *categorical product* $G \times H$ is a graph whose vertex set is $V(G) \times V(H)$ and any two vertices $(u, v)$ and $(u', v')$ are adjacent whenever $u$ is adjacent to $u'$ and $v$ is adjacent to $v'$. Hedetniemi's conjecture [10] is a very challenging and long-standing conjecture in graph theory which asserts that the chromatic number of the categorical product of two graphs is the minimum of their chromatic numbers. The chromatic number of the categorical product of graphs is studied extensively in the literature. For instance, one can refer to two extensive surveys by Tardif [25] and Zhu [26]. Hedetniemi's conjecture is open in general, although it has been verified for some families of graphs, especially those graphs which satisfy some topological conditions, see [11,16,22,24–26].

A hypergraph $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ is a family of nonempty subsets of $V(\mathcal{H})$. The elements of $V(\mathcal{H})$ and $E(\mathcal{H})$ are, respectively, called vertices and edges of $\mathcal{H}$. All hypergraphs considered in the paper have no multiple edges and $E(\mathcal{H})$ is thus a usual set. For a hypergraph $\mathcal{H}$, *the general Kneser graph* $\mathrm{KG}(\mathcal{H})$ is a graph with vertex set $E(\mathcal{H})$ such that two vertices are adjacent if their corresponding edges are vertex-disjoint. The coloring properties of general Kneser graphs were studied in many research articles, for instance [2,6,8,14,15,17,20]. It is simple to check that for any graph $G$, there are (infinitely many) hypergraphs $\mathcal{H}$ for which $G$ and $\mathrm{KG}(\mathcal{H})$ are isomorphic: for a graph $G$, let $V(\mathcal{H})$ be the union of $V(G)$ with the set of pairs of nonadjacent vertices of $G$, that is $V(\mathcal{H}) = V(G) \cup E(\bar{G})$, and for a vertex $u$ of $G$, the edge $e_u$ of $\mathcal{H}$ is the set consisting of the vertex $u$ and the pairs containing $u$. Such a hypergraph $\mathcal{H}$ is called *a Kneser representation of* $G$. *The Kneser graph* $\mathrm{KG}(n, k)$ can be defined in this way if we set $\mathcal{H} = ([n], \binom{[n]}{k})$, which justifies the name of Kneser representation. A set $I \subseteq V(\mathcal{H})$ is called *independent* if it contains no edge of $\mathcal{H}$.

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Let $V$ be a finite set. Define

$$L_V = \Big\{\sigma(1) < \cdots < \sigma(n) : \sigma : [n] \longrightarrow V \text{ is a bijection}\Big\}$$

to be the set of all linear orderings of $V = \{v_1, v_2, \ldots, v_n\}$. By abuse of language, when we write $\sigma = v_{i_1} < \cdots < v_{i_n}$, we simultaneously consider $\sigma$ as the linear ordering $v_{i_1} < \cdots < v_{i_n}$ and the bijective map $\sigma : [n] \longrightarrow V$ where $\sigma(j) = v_{i_j}$ for each $j$. Throughout the paper, we use this without any further mention. Let $\sigma : [n] \longrightarrow V$ be a linear ordering of $V$. A sequence $x_1 \ldots, x_q$ of vertices of $\mathcal{H}$ is said to be *in the order prescribed by* $\sigma$ if for each $l \in [q]$, $x_l = \sigma(j_l)$, where $j_1 < \cdots < j_q$. For two orderings $\sigma_1 = v_1 < \cdots < v_{m_1} \in L_V$ and $\sigma_2 = v'_1 < \cdots < v'_{m_2} \in L_{V'}$, by the concatenation of $\sigma_2$ after $\sigma_1$, denoted by $\sigma_1 \parallel \sigma_2$, we refer to the ordering $v_1 < \cdots < v_{m_1} < v'_1 < \cdots < v'_{m_2} \in L_{V \uplus V'}$.

Let $\mathcal{H} = (V, E)$ be a Kneser representation of $G$. The present authors [2] introduced the alternation number and strong alternation number of $\mathcal{H}$ and gave two lower bounds for the chromatic number of $G$ in terms of these two parameters. Given a linear ordering $\sigma$ of $V$ and a pair $\{I_1, I_2\}$ of independent sets of $\mathcal{H}$, the value of $\mathrm{alt}(\mathcal{H}, \sigma, \{I_1, I_2\})$ is the maximum length of a sequence (in the order prescribed by $\sigma$) of vertices of $\mathcal{H}$ alternating between membership in $I_1$ and $I_2$ (starting in any of the two sets). Similarly, for an independent set $I$, $\mathrm{salt}(\mathcal{H}, \sigma, I)$ is the maximum length of a sequence (in the order prescribed by $\sigma$) of vertices of $\mathcal{H}$ alternating between membership and nonmembership in $I$ (starting with membership or nonmembership). The value of $\mathrm{alt}(\mathcal{H}, \sigma)$ is the maximum of the values of $\mathrm{alt}(\mathcal{H}, \sigma, \{I_1, I_2\})$, and $\mathrm{alt}(\mathcal{H})$ is the minimum of the values of $\mathrm{alt}(\mathcal{H}, \sigma)$, where the maximum and minimum are, respectively, taken over all possible pairs $\{I_1, I_2\}$ and all linear orderings $\sigma \in L_V$. Similarly, $\mathrm{salt}(\mathcal{H}, \sigma)$ is the maximum of the values of $\mathrm{salt}(\mathcal{H}, \sigma, I)$, and $\mathrm{salt}(\mathcal{H})$ is the minimum of the values of $\mathrm{salt}(\mathcal{H}, \sigma)$, where the maximum and minimum are, respectively, taken over all independent sets $I$ and all linear orderings $\sigma \in L_V$. The two quantities $\mathrm{alt}(\mathcal{H})$ and $\mathrm{salt}(\mathcal{H})$ are, respectively, called *the alternation number* and *the strong alternation number of* $\mathcal{H}$. In other words, $\mathrm{alt}(\mathcal{H}, \sigma)$ (resp. $\mathrm{salt}(\mathcal{H}, \sigma)$) is the largest integer $k$ for which there is a subsequence (sub-ordering) $x_1 < x_2 \ldots < x_k$ of $\sigma = v_{i_1} < \cdots < v_{i_n}$ such that each (resp. at least one) of $\{x_j : j \text{ is odd}\}$ and $\{x_j : j \text{ is even}\}$ is an independent set of $\mathcal{H}$. It is clear that $\mathrm{alt}(\mathcal{H}, \sigma) \le \mathrm{salt}(\mathcal{H}, \sigma)$ and equality can hold. However, we can build a hypergraph $\mathcal{H}$ such that the difference between $\mathrm{alt}(\mathcal{H}, \sigma)$ and $\mathrm{salt}(\mathcal{H}, \sigma)$ is arbitrary large. To this end, let $\mathcal{H}_n$ be a hypergraph with vertex set $V(\mathcal{H}) = \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_n\}$ and the edge set $E(\mathcal{H}_n) = \Big\{\{v_i\} : i = 1, \ldots, n\Big\}$. For $\sigma = v_1 < u_1 < \cdots < v_n < u_n$, one can check that $\mathrm{alt}(\mathcal{H}, \sigma) = n$ while for $I = \{u_1, \ldots, u_n\}$, we have $\mathrm{salt}(\mathcal{H}, \sigma, I) = 2n$ which implies $\mathrm{salt}(\mathcal{H}, \sigma) = 2n$. Eventually, *the altermatic number* and the *strong altermatic number* of a graph $G$ are, respectively, defined as follows:

$$\zeta(G) = \max_{\mathcal{H}}\Big\{|V(\mathcal{H})| - \mathrm{alt}(\mathcal{H}) : \ \mathrm{KG}(\mathcal{H}) \longleftrightarrow G\Big\}$$

and

$$\zeta_s(G) = \max_{\mathcal{H}}\Big\{|V(\mathcal{H})| + 1 - \mathrm{salt}(\mathcal{H}) : \ \mathrm{KG}(\mathcal{H}) \longleftrightarrow G\Big\},$$

where $\mathrm{KG}(\mathcal{H}) \longleftrightarrow G$ means that $\mathrm{KG}(\mathcal{H})$ and $G$ are homomorphically equivalent in the sense that there are some homomorphism from $\mathrm{KG}(\mathcal{H})$ to $G$ and some homomorphism from $G$ to $\mathrm{KG}(\mathcal{H})$. For a graph $G$, a hypergraph $\mathcal{G}$ is called $\zeta$*-optimum* (resp. $\zeta_s$*-optimum*) *Kneser representation of* $G$ if $\mathrm{KG}(\mathcal{G}) \longleftrightarrow G$ and $\zeta(G) = |V(\mathcal{G})| - \mathrm{alt}(\mathcal{G})$ (resp. $\zeta_s(G) = 1 + |V(\mathcal{G})| - \mathrm{salt}(\mathcal{G})$). Such a hypergraph always exists since, in view of Theorem A, we have $\max\{\zeta(G), \zeta_s(G)\} < \infty$. Note that the aforementioned inequality $\mathrm{alt}(\mathcal{H}, \sigma) \le \mathrm{salt}(\mathcal{H}, \sigma)$ implies $\zeta(G) \ge \zeta_s(G) - 1$. Throughout the paper, we will frequently use $|V(\mathcal{H})| - \mathrm{alt}(\mathcal{H})$ and $|V(\mathcal{H})| - \mathrm{salt}(\mathcal{H}) + 1$, for simplicity of notation, we thus set $\overline{\mathrm{alt}}(\mathcal{H})$ and $\overline{\mathrm{salt}}(\mathcal{H})$ to refer to these two quantities, respectively.

The chromatic number of Kneser graphs $\mathrm{KG}(n, k)$ was computed by Lovász [17] solving a long standing conjecture by Kneser [12]. Lovász's proof makes use of algebraic topology giving birth to an area of combinatorics known as topological combinatorics. Later, Matoušek [19] found a purely combinatorial proof of Kneser's conjecture. However, his proof still has a topological flavor. Lovász's result was generalized to the general Kneser hypergraphs $\mathrm{KG}(\mathcal{H})$ by Dol'nikov [8]. Alishahi and Hajiabolhassan [2] used Tucker's lemma, a combinatorial counterpart of the Borsuk–Ulam theorem, and proved that both altermatic number and strong altermatic number provide sharp lower bounds for the chromatic number of graphs, see Theorem A. This result improves Dol'nikov's result as well as Lovász's result. Additionally, using this result, they were able to compute the chromatic number of some families of graphs, see [2–5]. It is also worth mentioning that Meunier [21] proved that for any fixed linear ordering $\sigma$, it is an NP-hard problem to compute $\mathrm{alt}(\mathcal{H}, \sigma)$ for a given hypergraph $\mathcal{H}$.

**Theorem A** (*[2]*)**.** *For any graph G, we have*

$$\chi(G) \ge \max\{\zeta(G), \zeta_s(G)\}.$$

Let $n$ and $k$ be two positive integers such that $n \ge 2k$. Note that if we set $\mathcal{H} = ([n], \binom{[n]}{k})$, then it is simple to see that $\overline{\mathrm{alt}}(\mathcal{H}) = \overline{\mathrm{salt}}(\mathcal{H}) = n - 2k + 2$ which implies that $\chi(\mathrm{KG}(n, k)) \ge \zeta(\mathrm{KG}(n, k)) \ge n - 2k + 2$ and $\chi(\mathrm{KG}(n, k)) \ge \zeta_s(\mathrm{KG}(n, k)) \ge n - 2k + 2$. Indeed, Kneser's conjecture states that $\chi(\mathrm{KG}(n, k)) \ge n - 2k + 2$. Consequently, Theorem A immediately implies Lovász's result.