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Note

Intersecting families, cross-intersecting families, and a proof of a conjecture of Feghali, Johnson and Thomas

Peter Borg

Department of Mathematics, Faculty of Science, University of Malta, Malta

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ABSTRACT

A family $\mathcal{A}$ of sets is said to be *intersecting* if every two sets in $\mathcal{A}$ intersect. Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$. For a positive integer n, let $[n] = \{1, \dots, n\}$ and $\mathcal{S}_n = \{A \subseteq [n] : 1 \in A\}$. We extend the Erdős–Ko–Rado Theorem by showing that if $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting families of subsets of $[n]$, $\mathcal{A}$ is intersecting, and $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$ are non-negative real numbers such that $a_i + b_i \geq a_{n-i} + b_{n-i}$ and $a_{n-i} \geq b_i$ for each $i \leq n/2$, then

$$\sum_{A\in\mathcal{A}} a_{|A|} + \sum_{B\in\mathcal{B}} b_{|B|} \leq \sum_{A\in\mathcal{S}_n} a_{|A|} + \sum_{B\in\mathcal{S}_n} b_{|B|}.$$

For a graph G and an integer $r \geq 1$, let $\mathcal{I}_G{}^{(r)}$ denote the family of r-element independent sets of G. Inspired by a problem of Holroyd and Talbot, Feghali, Johnson and Thomas conjectured that if $r < n$ and G is a *depth-two claw* with n leaves, then G has a vertex v such that $\{A \in \mathcal{I}_G{}^{(r)} : v \in A\}$ is a largest intersecting subfamily of $\mathcal{I}_G{}^{(r)}$. They proved this for $r \leq \frac{n+1}{2}$. We use the result above to prove the full conjecture.

1. Introduction

Unless otherwise stated, we shall use small letters such as x to denote non-negative integers or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose members are sets themselves). It is to be assumed that arbitrary sets and families are finite. We call a set A an *r-element set* if its size $|A|$ is r, that is, if it contains exactly r elements (also called members).

The set $\{1, 2, \dots\}$ of positive integers is denoted by $\mathbb{N}$. For any integer $n \geq 0$, the set $\{i \in \mathbb{N} : i \leq n\}$ is denoted by $[n]$. Note that $[0]$ is the empty set $\emptyset$. For a set X, the *power set of* X (that is, $\{A : A \subseteq X\}$) is denoted by 2^X. The family of r-element subsets of X is denoted by $\binom{X}{r}$. The family of r-element sets in a family $\mathcal{F}$ is denoted by $\mathcal{F}^{(r)}$. If $\mathcal{F} \subseteq 2^X$ and $x \in X$, then the family $\{F \in \mathcal{F} : x \in F\}$ is denoted by $\mathcal{F}(x)$ and called a *star of* $\mathcal{F}$.

We say that a set A *intersects* a set B if A and B have at least one common element (that is, $A \cap B \neq \emptyset$). A family $\mathcal{A}$ is said to be *intersecting* if for every $A, B \in \mathcal{A}$, A and B intersect. The stars of a family $\mathcal{F}$ (with $\bigcup_{F\in\mathcal{F}} F \neq \emptyset$) are the simplest intersecting subfamilies of $\mathcal{F}$. We say that $\mathcal{F}$ has the *star property* if at least one of the largest intersecting subfamilies of $\mathcal{F}$ is a star of $\mathcal{F}$.

One of the most popular endeavors in extremal set theory is that of determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$. This started in [18], which features the following classical result, known as the Erdős–Ko–Rado (EKR) Theorem.

Theorem 1.1 (*EKR Theorem [18]*). *If $r \leq n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$.*

E-mail address: peter.borg@um.edu.mt.

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This means that $\binom{[n]}{r}$ has the star property. There are various proofs of the EKR Theorem (see [14,29,31]), two of which are particularly short and beautiful: Katona's [31], which introduced the elegant cycle method, and Daykin's [14], using the fundamental Kruskal–Katona Theorem [30,32]. The EKR Theorem gave rise to some of the highlights in extremal set theory [1,20,29,34] and inspired many results that establish how large a system of sets can be under certain intersection conditions; see [6,15,21,22,23,26,27].

If $\mathcal{A}$ and $\mathcal{B}$ are families such that each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$, then $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting*.

For intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. A natural variant of this intersection problem is the problem of maximizing the sum or the product of sizes of cross-intersecting subfamilies (not necessarily distinct or non-empty) of $\mathcal{F}$. This has recently attracted much attention. The relation between the original intersection problem, the sum problem and the product problem is studied in [7]. Solutions have been obtained for various families; most of the known results are referenced in [8,9], which treat the product problem for families of subsets of $[n]$ of size at most r.

Here we consider the sum problem for the case where at least one of two cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $[n]$ is an intersecting family. We actually consider a more general setting of *weighted* sets, where each set of size i is assigned two non-negative integers a_i and b_i, and the objective is to maximize $\sum_{A\in\mathcal{A}} a_{|A|} + \sum_{B\in\mathcal{B}} b_{|B|}$. Note that $\sum_{A\in\mathcal{A}} a_{|A|} = |\mathcal{A}|$ if $a_0 = a_1 = \cdots = a_n = 1$. Let $\mathcal{S}_n$ denote the star $\{A \subseteq [n] : 1 \in A\}$ of $2^{[n]}$. In Section 2, we prove the following extension of the EKR Theorem.

Theorem 1.2. *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty cross-intersecting families of subsets of $[n]$, $\mathcal{A}$ is intersecting, and $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are non-negative real numbers such that $a_i + b_i \geq a_{n-i} + b_{n-i}$ and $a_{n-i} \geq b_i$ for each $i \leq n/2$, then*

$$\sum_{A\in\mathcal{A}} a_{|A|} + \sum_{B\in\mathcal{B}} b_{|B|} \leq \sum_{A\in\mathcal{S}_n} a_{|A|} + \sum_{B\in\mathcal{S}_n} b_{|B|}.$$

The EKR Theorem is obtained by taking $r \leq n/2$, $\mathcal{B} = \mathcal{A} \subseteq \binom{[n]}{r}$, and $b_i = 0 = a_i - 1$ for each $i \in \{0\} \cup [n]$.

We use Theorem 1.2 to prove a conjecture of Feghali, Johnson and Thomas [19, Conjecture 2.1]. Before stating the conjecture, we need some further definitions and notation.

A *graph* G is a pair $(X, \mathcal{Y})$, where X is a set, called the *vertex set of* G, and $\mathcal{Y}$ is a subset of $\binom{X}{2}$ and is called the *edge set of* G. The vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. An element of $V(G)$ is called a *vertex of* G, and an element of $E(G)$ is called an *edge of* G. We may represent an edge $\{v, w\}$ by vw. If vw is an edge of G, then we say that v is *adjacent* to w (in G). A subset I of $V(G)$ is an *independent set of* G if $\{v, w\} \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G$ denote the family of independent sets of G. An independent set J of G is *maximal* if $J \not\subseteq I$ for each independent set I of G such that $I \neq J$. The size of a smallest maximal independent set of G is denoted by $\mu(G)$.

Holroyd and Talbot introduced the problem of determining whether ${\mathcal{I}_G}^{(r)}$ has the star property for a given graph G and an integer $r \geq 1$. The Holroyd–Talbot (HT) Conjecture [27, Conjecture 7] claims that ${\mathcal{I}_G}^{(r)}$ has the star property if $\mu(G) \geq 2r$. The author [4] proved that the conjecture is true if $\mu(G)$ is sufficiently large depending on r (see also [11, Lemma 4.4 and Theorem 1.4]). By the EKR Theorem, the conjecture is true if G has no edges. The HT Conjecture has been verified for several classes of graphs [12,13,24–28,33,35]. As demonstrated in [13], for $r > \mu(G)/2$, whether ${\mathcal{I}_G}^{(r)}$ has the star property or not depends on G and r (both cases are possible).

A *depth-two claw* is a graph consisting of n pairwise disjoint edges $x_1y_1, \ldots, x_ny_n$ together with a vertex $x_0 \notin \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ that is adjacent to each of $y_1, \ldots, y_n$. This graph will be denoted by T_n. Thus, $T_n = (\{x_0, x_1, \ldots, x_n, y_1, \ldots, y_n\}, \{x_0y_1, \ldots, x_0y_n, x_1y_1, \ldots, x_ny_n\})$. For each $i \in [n]$, we may take x_i and y_i to be $(i, 1)$ and $(i, 2)$, respectively. Let $X_n = \{x_i : i \in [n]\}$ and

$$\mathcal{L}_n = \left\{\{(i_1, j_1), \ldots, (i_r, j_r)\} : r \in [n], \{i_1, \ldots, i_r\} \in \binom{[n]}{r}, j_1, \ldots, j_r \in \{1, 2\}\right\}.$$

Note that

$${\mathcal{I}_{T_n}}^{(r)} = {\mathcal{L}_n}^{(r)} \cup \left\{A \cup \{x_0\} : A \in \binom{X_n}{r-1}\right\}. \quad (1)$$

The family ${\mathcal{I}_{T_n}}^{(r)}$ is empty for $r > n+1$, and consists only of the set $\{x_0, x_1, \ldots, x_n\}$ for $r = n+1$. In [19], Feghali, Johnson and Thomas showed that ${\mathcal{I}_{T_n}}^{(r)}$ does not have the star property for $r = n$, and they made the following conjecture.

Conjecture 1.3 ([19]). *If $1 \leq r \leq n-1$, then ${\mathcal{I}_{T_n}}^{(r)}$ has the star property.*

They proved the conjecture for $r \leq \frac{n+1}{2}$.

Theorem 1.4 ([19]). *If $1 \leq r \leq \frac{n+1}{2}$, then ${\mathcal{I}_{T_n}}^{(r)}$ has the star property.*

In the next section, we settle the full conjecture, using Theorem 1.2 for $r > \frac{n+1}{2}$.