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The pentavalent three-geodesic-transitive graphs[☆]

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ABSTRACT

A complete classification is given of pentavalent 3-geodesic-transitive graphs which are not 3-arc-transitive, which shows that a pentavalent 3-geodesic-transitive but not 3-arc-transitive graph is one of the following graphs: $\overline{(2\times 6)\text{-grid}}$, H(5, 2), the icosahedron, the incidence graph of the 2-(11, 5, 2)-design, the Wells graph and the Sylvester graph.

1. Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [3,8,23]. A *geodesic* from a vertex u to a vertex v in a graph Γ is one of the shortest paths from u to v in Γ, and this geodesic is called an *s-geodesic* if the distance between u and v is s. Then Γ is said to be *s-geodesic transitive* if, Γ has an s-geodesic, and for each $1\leq i\leq s$, the automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the set of i-geodesics of Γ. For a positive integer s, an s-arc of Γ is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ in Γ such that v_i, v_{i+1} are adjacent and $v_{j-1}\neq v_{j+1}$ where $0\leq i\leq s-1$ and $1\leq j\leq s-1$. In particular, 1-arcs are called *arcs*. Then Γ is said to be *s-arc transitive* if, for each $i\leq s$, the group $\mathrm{Aut}(\Gamma)$ is transitive on the set of i-arcs of Γ. Thus a graph is s-geodesic transitive (s-arc transitive), then it is t-geodesic transitive (t-arc transitive) for each $t\leq s$.

Clearly, every s-geodesic is an s-arc, but some s-arcs may not be s-geodesics whenever $s\geq 2$. If Γ has girth 3 (the girth of Γ, denoted by *girth*(Γ), is the length of the shortest cycle in Γ), then the 3-arcs contained in a triangle are not 3-geodesics. The graph in Fig. 1 is the icosahedron, which is 3-geodesic-transitive but not 3-arc-transitive with valency 5 and girth 3. Thus the family of 3-arc-transitive graphs is properly contained in the family of 3-geodesic-transitive graphs.

The first remarkable result about s-arc transitive graphs comes from Tutte [20,21], and this family of graphs has been studied extensively, see [1,14,16,19,22,24]. The local structure of the family of 2-geodesic-transitive graphs was determined in [5]. In [4], Devillers, Li, Praeger and the author classified 2-geodesic-transitive graphs of valency 4. Later, in [6], a reduction theorem for the family of normal 2-geodesic-transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. The family of 2-geodesic-transitive but not 2-arc-transitive graphs with prime valency was precisely determined in [7]. In [15], the author classified the family of 3-geodesic-transitive but not 3-arc-transitive graphs of valency 4. Li and Feng [17] studied the family of pentavalent 1-regular graphs of square free order. Following this, Hua et al. [13] classified pentavalent symmetric graphs of order $2pq$. Guo and Feng [10] determined the stabilizers of pentavalent symmetric graphs. In [18], Pan, Luo and Liu gave a classification of arc-transitive pentavalent graphs

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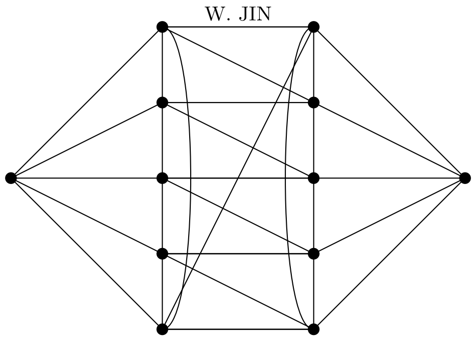

Fig. 1. Icosahedron.

of order $4pq$, where $p, q \geq 5$ are distinct primes. Recently, Du et al. [9] investigated pentavalent symmetric graphs admitting vertex-transitive non-abelian simple groups. Despite all of these efforts, however, further classifications of symmetric graphs with valency 5 seem to be very difficult, it has been considered for many years by several authors, but it still has not been achieved.

The purpose of this paper is to give a complete classification of the family of pentavalent 3-geodesic-transitive but not 3-arc-transitive graphs, stated in the following main theorem.

Theorem 1.1. *Let Γ be a connected pentavalent 3-geodesic-transitive but not 3-arc-transitive graph. Then Γ is one of the following graphs:* $\overline{(2 \times 6)\text{-grid}}$, $\mathrm{H}(5, 2)$*, the incidence graph of the* 2-(11, 5, 2)*-design, the icosahedron, the Wells graph or the Sylvester graph.*

The *Hamming graph* $\mathrm{H}(5, 2)$ has vertex set $\Delta^5 = \{(x_1, x_2, x_3, x_4, x_5) | x_i \in \Delta\}$, the cartesian product of 5-copies of Δ, where $\Delta = \{1, a\}$, and two vertices v and v' are adjacent if and only if they are different in exactly one coordinate. For $m, n \geq 2$, the $(m \times n)$-grid is the graph with vertex set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, and 2 distinct vertices (i, j) and (r, s) are adjacent if and only if $i = r$ or $j = s$. The $(m \times n)$-grid is also the line graph of the complete bipartite graph $\mathrm{K}_{m,n}$, and so its automorphism group is $S_m \times S_n$ when $m \neq n$ and $S_m \wr S_2$ when $m = n$. For a graph Γ, its *complement* $\overline{\Gamma}$ is the graph with vertex set $V(\Gamma)$, and two vertices are adjacent if and only if they are not adjacent in Γ.

The 2-(11, 5, 2)-*design* is a pair $\mathcal{D} = \{\mathcal{X}, \mathcal{B}\}$, where $\mathcal{X}$ is a set of points of cardinality 11, and $\mathcal{B}$ a set of 5-subsets of $\mathcal{X}$ called *blocks*, with the property that every 2 points are contained in exactly 2 blocks. The incidence graph of $\mathcal{D}$ is a bipartite graph with two parts $\mathcal{X}$ and $\mathcal{B}$ such that a point x of $\mathcal{X}$ is adjacent to a block b of $\mathcal{B}$ if and only if x is contained in b, and this graph has automorphism group M_{12} with 22 vertices, see [2, p. 227] and [12]. The *Wells graph* (also known as the *Armanios–Wells graph*) is an antipodal graph of 32 vertices and diameter 4, and it is the unique double cover without 4-cycles of the folded 5-cube. The automorphism group of the Wells graph is $\mathbb{Z}_2^{1+4} : A_5$, where $\mathbb{Z}_2^{1+4} = D_8 \circ Q_8$ is an extra-special group of order 32 and A_5 is the vertex stabilizer, refer to [2, p. 266]. The *Sylvester graph* is the graph on the 36 pairs (ovoid,spread) in $GQ(2, 2)$, where (O, S) is adjacent to (O', S') when the unique point in both O and O' lies on the unique line in both S and S'. This graph has 36 vertices with automorphism group $\mathrm{Aut}(S_6)$ and vertex-stabilizer $AGL(1, 5) \times \mathbb{Z}_2$, see Section 13.1 A of [2].

Finally, we pose a problem. The family of 2-geodesic-transitive but not 2-arc-transitive graphs with prime valency was precisely determined in [7]. Thus the following problem is interesting to be investigated.

Problem 1.2. Classify the family of 3-geodesic-transitive but not 3-arc-transitive graphs of prime valency.

2. Proof of Theorem 1.1

We prove our main theorem by a series of lemmas. In the characterization of 3-geodesic-transitive graphs, the following constants are useful. Our definition is inspired by the concept of intersection arrays defined for the distance-regular graphs (see [2]).

Definition 2.1. Let Γ be an s-geodesic-transitive graph, $u \in V(\Gamma)$, and let $v \in \Gamma_i(u)$, $i \leq s$. Then the number of edges from v to $\Gamma_{i-1}(u)$, $\Gamma_i(u)$, and $\Gamma_{i+1}(u)$ does not depend on the choice of v and these numbers are denoted, respectively, by c_i, a_i, b_i.

Clearly we have that $a_i + b_i + c_i$ is equal to the valency of Γ whenever the constants are well-defined. Note that for 3-geodesic-transitive graphs, the constants are always well-defined for $i = 1, 2, 3$.

A subgraph X of a graph Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U. The *diameter* $\mathrm{diam}(\Gamma)$ of a graph Γ is the maximum