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Conditions for graphs to be path partition optimal[☆]

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ABSTRACT

The path partition number of a graph is the minimum number of edges we have to add to turn it into a Hamiltonian graph, and the separable degree is the minimum number of edges we have to add to turn it into a 2-connected graph. A graph is called path partition optimal if its path partition number is equal to its separable degree. We study conditions that guarantee path partition optimality. We extend several known results on Hamiltonicity to path partition optimality, in particular results involving degree conditions and induced subgraph conditions.

1. Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph. We use $n(G)$, $e(G)$ and $c(G)$ to denote the numbers of vertices, edges and components of G, respectively.

A *path partition* of G is a spanning subgraph of G each component of which is a path of G. If G is a non-Hamiltonian graph, then the *path partition number* of G, denoted by $\pi(G)$, is the minimum number of components in a path partition of G; if G is Hamiltonian, then we define $\pi(G) = 0$. Alternatively, $\pi(G)$ is the minimum number of edges we have to add to G to turn it into a Hamiltonian graph, except for degenerate cases. Note that $\pi(K_1) = \pi(K_2) = 1$ and $\pi(2K_1) = 2$.

The *separable degree* of G, denoted by $\sigma(G)$, is defined as the minimum number of edges we need to add to G to turn it into a 2-connected graph, again except for degenerate cases. We define $\sigma(K_1) = \sigma(K_2) = 1$ and $\sigma(2K_1) = 2$. Note that every 2-connected graph has separable degree 0 and every disconnected graph has separable degree at least 2.

The following proposition is obvious.

Proposition 1. *For every graph G, $\pi(G) \geq \sigma(G)$.*

Proof. If G has only one or two vertices, then the result is trivially true. If G has at least three vertices, then the result can be obtained from the fact that a Hamiltonian graph is necessarily 2-connected. □

In this paper, we are interested in conditions on G that imply $\pi(G) = \sigma(G)$. For this purpose, we say that a graph G with property $\pi(G) = \sigma(G)$ is *path partition optimal*. It is clear from the above definitions that K_1, K_2 and $2K_1$ are path partition

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optimal. In the following sections, we consider degree and subgraph conditions for path partition optimality of graphs. Since it is necessary to introduce a lot of terminology and notation in order to state our results, we refrain from stating these results explicitly in this introductory section. We apologize for the inconvenience this may cause.

The rest of the paper is organized as follows. In Section 2, we extend Chvátal–Erdős' Theorem and Ore's Theorem for Hamiltonicity, and obtain a degree sum condition for path partition optimality. In Section 3, we present extensions of known forbidden induced subgraph results for Hamiltonicity. These extensions are relatively easy to prove. Our main result is presented in Section 4, together with the necessary background. It is an extension of known heavy subgraph results for Hamiltonicity. It is also an extension of both the degree sum conditions and forbidden subgraph conditions for path partition optimality that we present in Sections 2 and 3. This extension is more difficult to prove, as we will explain and illustrate in Section 4. Most of the terminology and auxiliary results that we need for the proof of our main result are presented in Section 5, followed by our proof in Section 6.

2. Degree sum conditions

We first prove easy extensions of Ore's Theorem and Chvátal–Erdős' Theorem for Hamiltonicity.

For a graph G, let $\kappa(G)$ and $\alpha(G)$ denote the connectivity and independence number of G, respectively. If G is not complete, we define $s(G)$ as the minimum degree sum of any two nonadjacent vertices of G. If G is complete, then we define $s(G) = \infty$. We recall Ore's Theorem and Chvátal–Erdős' Theorem for convenience.

Theorem 1 (*Ore [11]*)**.** *Let G be a 2-connected graph. If $s(G) \geq n(G)$, then G is Hamiltonian.*

Theorem 2 (*Chvátal and Erdős [3]*)**.** *Let G be a graph of order at least 3. If $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

Bondy [1] proved that any simple graph satisfying Ore's condition also satisfies Chvátal–Erdős' condition. Thus, in this sense Theorem 2 is a generalization of Theorem 1. Recall that a graph G is Hamiltonian if and only if $\pi(G) = 0$. As an extension of Theorems 1 and 2, we obtain the following result.

Theorem 3. *Let G be a graph.*

(1) *If $\alpha(G) \leq \kappa(G) + \sigma(G)$, then G is path partition optimal.*
(2) *If $s(G) \geq n(G) - \sigma(G)$, then $\alpha(G) \leq \kappa(G) + \sigma(G)$, and thus G is path partition optimal.*

Proof. (1) If $n(G) \leq 2$, then the result is trivially true. So we assume that $n(G) \geq 3$. If G is 2-connected, then the result follows directly from Theorem 2. So we assume that $\sigma(G) \geq 1$.

Let Y be a set of vertices such that $Y \cap V(G) = \emptyset$ and $|Y| = \sigma(G)$. We construct a graph G' such that $V(G') = V(G) \cup Y$ and $E(G') = E(G) \cup \{uv : u \in Y, v \in V(G'), u \neq v\}$. Note that Y is a clique in G' and that every vertex of Y is adjacent to every vertex of $V(G)$ in G'. It follows that $\kappa(G') = \kappa(G) + \sigma(G)$ and $\alpha(G') = \alpha(G)$. Thus $\alpha(G') \leq \kappa(G')$. By Theorem 2, G' is Hamiltonian.

Let C be a Hamilton cycle of G'. Then $C - Y$ is a path partition of G with at most $\sigma(G)$ paths. This implies that $\pi(G) \leq \sigma(G)$. By Proposition 1, $\pi(G) \geq \sigma(G)$. Thus G is path partition optimal.

(2) If $\sigma(G) = 0$, i.e., G is 2-connected, then it follows from Bondy's result [1] that $\alpha(G) \leq \kappa(G)$. So we assume that $\sigma(G) \geq 1$.

Let Y and G' be defined as above. Note that $d_{G'}(v) = d_G(v) + \sigma(G)$ for every $v \in V(G)$. We have $n(G') = n(G) + \sigma(G)$ and $s(G') = s(G) + 2\sigma(G)$. This implies that $s(G') \geq n(G')$. Thus $\alpha(G') \leq \kappa(G')$. Also recall that $\kappa(G') = \kappa(G) + \sigma(G)$ and $\alpha(G') = \alpha(G)$. Thus we have $\alpha(G) \leq \kappa(G) + \sigma(G)$. By (1), G is path partition optimal. □

3. Forbidden subgraph conditions

In this section, we prove extensions of known Hamiltonicity results involving forbidden (induced) subgraph conditions. It turns out that these results are easy to prove, contrary to the main result of Section 4, involving heavy subgraph conditions. We will give a short explanation for this difference in Section 4.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G. For a given graph H, we say that G is *H-free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

Note that a K_2-free graph is an empty (edgeless) graph and that a $2K_1$-free graph is a complete graph. In order to avoid a discussion of the trivial cases, in the following, when we say that H is an induced subgraph of a graph G or that G is an H-free graph, we always assume that H is of order at least 3.

We say that a graph G is *traceable* if G has a *Hamilton path*, i.e., a path containing all its vertices. If a graph G is connected and P_3-free (where P_3 is the path on three vertices), then it is a complete graph; in that case, G is traceable (and Hamiltonian if $n(G) \geq 3$). In fact, it is well-known and folklore that P_3 is the only subgraph with this property.

If a disconnected graph G is P_3-free, then every component of G is a clique. Clearly, for such a graph $\sigma(G) = c(G) = \pi(G)$. After checking the trivial cases, it is easy to show that every P_3-free graph is path partition optimal. In fact, we have the following characterization.