Discrete Mathematics 341 (2018) 875–888

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cores with distinct parts and bigraded Fibonacci numbers

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ARTICLE INFO

Article history:
Received 21 June 2017
Received in revised form 8 December 2017
Accepted 14 December 2017
Available online 30 January 2018

Keywords:
Core partition
Dyck path
Abacus diagram
Fibonacci numbers

ABSTRACT

The notion of (a, b)-cores is closely related to rational (a, b)-Dyck paths via the bijection due to J. Anderson, and thus the number of $(a, a+1)$-cores is given by the Catalan number C_a. Recent research shows that $(a, a+1)$-cores with distinct parts are enumerated by another important sequence— Fibonacci numbers F_a. In this paper, we consider the abacus description of (a, b)-cores to introduce the natural grading and generalize this result to $(a, as+1)$-cores. We also use the bijection with Dyck paths to count the number of $(2k-1, 2k+1)$-cores with distinct parts. We give a second grading to Fibonacci numbers, induced by the bigraded Catalan sequence $C_{a,b}(q, t)$.

1. Introduction

For two coprime integers a and b, the rational Catalan number $C_{a,b}$ and its bigraded generalization $C_{a,b}(q, t)$ have caught the attention of different researchers due to their connection to algebraic combinatorics and geometry [4,5,7,8]. Catalan numbers can be analyzed from the perspective of different combinatorial objects: rational (a, b)-Dyck paths, simultaneous (a, b)-core partitions and abacus diagrams.

In 2015, Amdeberhan [1] conjectured that the number of $(a, a+1)$-cores with distinct parts is equal to the Fibonacci number F_{a+1}, and also conjectured the formulas for the largest size and the average size of such partitions. This conjecture has been proven by Xiong:

Theorem 1 (*Xiong, [14]*). *For $(a, a+1)$-core partitions with distinct parts, we have*

(1) *the number of such partitions is F_{a+1};*
(2) *the largest size of such partition is $\left\lfloor \frac{1}{3}\binom{a+1}{2} \right\rfloor$;*
(3) *there are $\frac{3-(-1)^{a \bmod 3}}{2}$ such partitions of maximal size;*
(4) *the total number of these partitions and the average sizes are, respectively, given by*

$$\sum_{i+j+k=a+1} F_i F_j F_k \quad \text{and} \quad \sum_{i+j+k=a+1} \frac{F_i F_j F_k}{F_{a+1}}.$$

Part (1) of the above theorem was independently proved by Straub [13].

Another interesting conjecture of Amdeberhan is the number of $(2k-1, 2k+1)$-cores with distinct parts. This conjecture has been proven by Yan, Qin, Jin and Zhou:

Theorem 2 (*YQJZ, [16]*). *The number of $(2k-1, 2k+1)$-cores with distinct parts is equal to 2^{2k-2}.*

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https://doi.org/10.1016/j.disc.2017.12.010

The proof uses somewhat complicated arguments about the poset structure of cores. Results by Zaleski and Zeilberger [18] improve the argument using Experimental Mathematics tools in Maple. More recently Baek, Nam and Yu provided a simpler bijective proof in [6].

Another set of combinatorial objects that has caught the attention of a number of researchers [12,13,15,17] is the set of $(a, as \pm 1)$-cores with distinct parts. In particular, there is a Fibonacci-like recursive relation for the number of such cores:

Theorem 3 (*Straub, [13]*)**.** *The number $E_s^-(a)$ of $(a, as-1)$-core partitions with distinct parts is characterized by $E_s^-(1) = 1$, $E_s^-(2) = s$ and, for $a \geq 3$,*

$$E_s^-(a) = E_s^-(a-1) + sE_s^-(a-2).$$

Theorem 4 (*Nath and Sellers, [12]*)**.** *The number $E_s^+(a)$ of $(a, as+1)$-core partitions with distinct parts is characterized by $E_s^+(1) = 1$, $E_s^+(2) = s+1$ and, for $a \geq 3$,*

$$E_s^+(a) = E_s^+(a-1) + sE_s^+(a-2).$$

In this paper, we analyze simultaneous core partitions in the context of Anderson's bijection and in Section 3 we provide a simple description of the set of $(a, as+1)$-cores with distinct parts in terms of abacus diagrams, which also allows us to provide another proof of Theorem 1 parts (1), (2) and (3) in Section 4.

In Section 5 we use the connection between cores and Dyck paths to provide another simple proof of Theorem 2.

In Section 6 we introduce graded Fibonacci numbers

$$F_{a,b}(q) = \sum_{\kappa} q^{area(\kappa)},$$

where the sum is taken over all (a, b)-cores κ with distinct parts and *area* is some statistic on (a, b)-cores. We show that $F_{a,a+1}(1) = F_{a+1}$-the regular Fibonacci sequence, and prove recursive relations for $F_a^{(s)}(q) := F_{a,as+1}(q)$. Using properties of $F_{a,a+1}(q)$ we provide another proof of Theorem 4 and another proof of Theorem 1 part (4).

In Section 7 we introduce bigraded Fibonacci number as a summand of bigraded Catalan numbers:

$$F_a^{(s)}(q, t) = \sum_{\pi} q^{area(\pi)} t^{bounce(\pi)},$$

where the sum is taken over all $(a, as+1)$-Dyck paths corresponding to $(a, as+1)$-cores with distinct parts, and statistics $(area, bounce)$ are two standard statistics on Dyck paths (see [11]).

Using abacus diagrams, we can get a simple formula for $F_a^{(s)}(q, t)$ and prove a theorem that gives recursive relations similar to the recursive relations for regular Fibonacci numbers. We use the standard notation $(s)_r = 1 + r + \cdots + r^{s-1}$.

Theorem 5. *Normalized bigraded Fibonacci numbers $\tilde{F}_a^{(s)}(q, t)$ satisfy the recursive relations*

$$\tilde{F}_{a+1}^{(s)}(q, t) = \tilde{F}_a^{(s)}(q, t) + qt^a (s)_{qt^a} \tilde{F}_{a-1}^{(s)}(q, t) = \tilde{F}_a^{(s)}(qt, t) + qt(s)_{qt} \tilde{F}_{a-1}^{(s)}(qt^2, t),$$

with initial conditions $\tilde{F}_0^{(s)}(q, t) = \tilde{F}_1^{(s)}(q, t) = 1$.

2. Background and notation

For two coprime numbers a and b consider a rectangle $R_{a,b}$ on the square lattice with bottom-left corner at the origin and top-right corner at (a, b). We call the diagonal from $(0, 0)$ to (a, b) the *main diagonal* of the rectangle $R_{a,b}$. An (a, b)-*Dyck path* is a lattice path from $(0, 0)$ to (a, b) that consists of North and East steps and that lies weakly above the main diagonal. Denote the set of (a, b)-Dyck paths by $\mathrm{D}_{a,b}$.

For a box in $R_{a,b}$ with bottom-right corner coordinates (x, y), define the *rank of the box* to be equal to $ay - bx$ (see Fig. 1, left). Note that a box has positive rank if and only if it lies above the main diagonal. For a rational Dyck path π, we define the area statistic $area(\pi)$ to be the number of boxes in $R_{a,b}$ with positive ranks that are below π.

Denote the set of ranks of all the area boxes of π as $\alpha(\pi)$. Note that $\alpha(\pi)$ does not contain any multiples of a or b and it has an (a, b)-*nested property*, that is,

$$(i \in \alpha(\pi),\ i > a) \Rightarrow i - a \in \alpha(\pi), \quad (j \in \alpha(\pi),\ j > b) \Rightarrow j - b \in \alpha(\pi). \tag{1}$$

Also note that $\alpha(\pi)$ completely determines the Dyck path π.

Remark 6. The (a, b)-nested property of $\alpha(\pi)$ is equivalent to the (a, b)-invariant property of the complement of $\alpha(\pi)$ (see [8]).