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Graphs vertex-partitionable into strong cliques

Ademir Hujdurović [a,b], Martin Milanič [a,b,*], Bernard Ries [c]

[a] University of Primorska, IAM, Muzejski trg 2, SI-6000 Koper, Slovenia
[b] University of Primorska, FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia
[c] University of Fribourg, Department of Informatics, Bd de Pérolles 90, CH-1700 Fribourg, Switzerland

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ABSTRACT

A clique in a graph is strong if it intersects all maximal independent sets. A graph is localizable if it has a partition of the vertex set into strong cliques. Localizable graphs were introduced by Yamashita and Kameda in 1999 and form a rich class of well-covered graphs that coincides with the class of well-covered graphs within the class of perfect graphs. In this paper, we give several equivalent formulations of the property of localizability and develop polynomially testable characterizations of localizable graphs within three non-perfect graph classes: triangle-free graphs, C_4-free graphs, and line graphs. Furthermore, we use localizable graphs to construct an infinite family of counterexamples to a conjecture due to Zaare-Nahandi about k-partite well-covered graphs having all maximal cliques of size k.

1. Introduction

1.1. Background

A *clique* (resp., *independent set*) in a graph is a set of pairwise adjacent (resp., pairwise non-adjacent) vertices. A clique (resp., independent set) in a graph is said to be *maximal* if it is not contained in any larger clique (resp., independent set), and *strong* if it intersects every maximal independent set (resp., every maximal clique). In other words, a strong clique is a transversal of the family of all maximal independent sets of the graph (in fact, an *exact* transversal, since a clique can intersect an independent set in at most one vertex). The notions of strong cliques and strong independent sets in graphs have given rise to several interesting graph classes studied in the literature, including strongly perfect graphs [3,4], very strongly perfect graphs [10,33,45], general partition graphs [38,44], and CIS graphs [6,19] (see also [7]). It is co-NP-complete to test whether a given clique in a graph is strong [65] and NP-hard to test whether a given graph contains a strong clique [34]. On the other hand, the complexity of recognizing: (i) strongly perfect, (ii) general partition, and (iii) CIS graphs is, to the best of our knowledge, open. In terms of strong cliques, these problems are equivalent to testing if: (i) every induced subgraph of a given graph has a strong clique, (ii) every edge of a given graph is contained in a strong clique, and (iii) every maximal clique in a given graph is strong.

In this paper we study two interrelated graph classes, the class of well-covered graphs and the class of localizable graphs. A graph is *well-covered* if all its maximal independent sets have the same size and *localizable* if it has a partition of the vertex set into strong cliques. Well-covered graphs were introduced by Plummer in 1970 [50] and have been studied extensively in the literature, for various reasons. A most obvious reason is that the maximum independent set problem, which is generally NP-complete, can be solved in polynomial time in the class of well-covered graphs by a greedy algorithm.

* Corresponding author at: University of Primorska, FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia.
E-mail addresses: ademir.hujdurovic@upr.si (A. Hujdurović), martin.milanic@upr.si (M. Milanič), bernard.ries@unifr.ch (B. Ries).

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Furthermore, well-covered graphs have applications in distributed systems [63], are related to the so-called Generalized Kayles game [22,24], and play an important role in commutative algebra, where they are typically referred to as *unmixed* graphs, see, e.g., [32,47,61]. The well-coveredness property of a graph G is equivalent to the property that the simplicial complex of the independent sets of G is *pure* and generalizes the algebraically defined concept of a Cohen–Macaulay graph (see, e.g., [13]). For further background on well-covered graphs, we refer to the surveys by Plummer [51] and Hartnell [30].

It is not difficult to see that every localizable graph is well-covered. However, unlike well-covered graphs, localizable graphs have not been much studied in the literature. They were introduced in 1999 by Yamashita and Kameda [63] and appeared, at least implicitly, in other works, for example in 1983 in the work of Finbow and Hartnell [24, Proposition 1] and more recently, in 2015, the work of Zaare-Nahandi [64, Theorem 2.1]. In particular, Finbow and Hartnell proved in [24] that every well-covered graph of girth at least 8 has a perfect matching formed by pendant edges. Since every pendant edge is a strong clique, this implies that every well-covered graph of girth at least 8 is localizable. Unaware of this result, Yamashita and Kameda observed in [63] that all well-covered trees are localizable, pointed out that the converse inclusion fails in general, and asked for a characterization of localizable graphs.

1.2. Aims and motivation

In this work we initiate a systematic study of localizable graphs. Our motivations for this study are threefold.

First, the class of localizable graphs is defined via the notion of strong cliques in graphs, which, as outlined above, gives rise to many interesting and challenging graph classes. It is also worth pointing out that strong cliques can arise as images, under suitable transformations, of other well-studied objects in graphs; for example, transforming a regular triangle-free graph to the complement of its line graph maps any perfect matching into a strong clique (see [5,46] for applications of this observation). All these facts lead us to believe that the class of localizable graphs is worthy of study on its own and that investigating it is likely to lead to interesting new results and questions on connections between various graph notions, inclusions between graph classes, and complexity considerations. Indeed, some of the characterizations obtained in this paper unify and generalize several known results from the literature. Furthermore, localizable graphs will be a useful tool in our construction of counterexamples to a conjecture due to Zaare-Nahandi [64] about k-partite well-covered graphs having all maximal cliques of size k.

Second, a better understanding of the class of localizable graphs may lead to further insights on the class of well-covered graphs. Since every localizable graph is well-covered, any construction leading to localizable graphs will immediately result in a family of well-covered graphs. Moreover, as we will see in Section 2, the two classes coincide within the class of perfect graphs, and, more generally, within the class of semi-perfect graphs, defined as graphs in which the clique cover number and the independence number coincide [64]. This relation may lead to new perspectives and insights into classes of well-covered perfect and semi-perfect graphs.

Finally, the study of localizable graphs directly addresses the question about characterizations of localizable graphs raised by Yamashita and Kameda. As we will show in Section 2, it is generally NP-hard to determine if a given graph is localizable. This motivates the question of identifying graph classes where the property of localizability can be tested efficiently. Several examples of such graph classes will be presented in this paper.

1.3. Overview of results

Our results can be roughly divided into three main parts, which we now summarize.

1. Equivalent formulations. We give several equivalent formulations of localizability, which allow us to derive several characterizations of localizable graphs within the class of semi-perfect graphs.

2. Counterexamples to a conjecture due to Zaare-Nahandi. We use localizable graphs to construct an infinite family of counterexamples to a conjecture due to Zaare-Nahandi about k-partite well-covered graphs having all maximal cliques of size k.

Conjecture 1 (*Zaare-Nahandi [64]*)**.** *Let G be a k-partite well-covered graph in which all maximal cliques are of size k. Then G is semi-perfect.*

A graph is *co-well-covered* if its complement is well-covered. We will show that Conjecture 1 can be equivalently posed in terms of localizable graphs, as follows.

Conjecture 2. *Let G be a localizable co-well-covered graph. Then $\overline{G}$ is localizable.*

We disprove these two equivalent conjectures by constructing an infinite family of counterexamples to the weaker statement saying that every localizable co-well-covered graph has a strong independent set. We also give a related hardness result showing that it is NP-hard to determine whether the complement of a given localizable co-well-covered graph is localizable. The proof is based on a reduction from the 3-colorability problem in triangle-free graphs and shows a way how to transform, in a simple way, any triangle-free graph of chromatic number at least four to a counterexample to Conjecture 1.

3. Characterizations. We characterize localizable graphs within the classes of triangle-free graphs and C_4-free graphs. Both characterizations lead to polynomial-time recognition algorithms. To put these results in perspective, note that no