Discrete Mathematics 341 (2018) 889–895

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

More circulant graphs exhibiting pretty good state transfer

Hiranmoy Pal

Department of Mathematics, Indian Institute of Information Technology Bhagalpur, Bihar, 813210, India

ARTICLE INFO

Article history:
Received 27 May 2017
Received in revised form 2 November 2017
Accepted 18 December 2017
Available online 17 January 2018

Keywords:
Graph
Circulant graph
Pretty good state transfer

ABSTRACT

The transition matrix of a graph G corresponding to the adjacency matrix A is defined by $H(t) := \exp(-itA)$, where $t \in \mathbb{R}$. The graph is said to exhibit pretty good state transfer between a pair of vertices u and v if there exists a sequence $\{t_k\}$ of real numbers such that $\lim_{k\to\infty} H(t_k)\mathbf{e}_u = \gamma\mathbf{e}_v$, where γ is a complex number of unit modulus. We present a class of circulant graphs admitting pretty good state transfer. Also we find some circulant graphs not exhibiting pretty good state transfer. This generalizes several pre-existing results on circulant graphs admitting pretty good state transfer.

1. Introduction

In [9], Godsil first introduced the notion of pretty good state transfer (PGST) in the context of continuous-time quantum walks. State transfer has significant applications in quantum information processing and cryptography (see [4,6]). We consider state transfer with respect to the adjacency matrix of a graph. It can be considered with respect the Laplacian matrix as well. However, for a regular graph G, one can observe that G admits PGST with respect to the adjacency matrix if and only if G admits PGST with respect to the Laplacian matrix. All graphs we consider are simple, undirected and finite. Now we define PGST on a graph G with adjacency matrix A. The transition matrix of G is defined by

$$H(t) := \exp(-itA), \text{ where } t \in \mathbb{R}.$$

Let $\mathbf{e}_u$ denote the characteristic vector corresponding to the vertex u of G. The graph G is said to exhibit PGST between a pair of vertices u and v if there is a sequence $\{t_k\}$ of real numbers such that

$$\lim_{k\to\infty} H(t_k)\mathbf{e}_u = \gamma\mathbf{e}_v, \text{ where } \gamma \in \mathbb{C} \text{ and } |\gamma| = 1.$$

In the past decade the study of state transfer has received considerable attention. Now we know a few graphs exhibiting PGST. The main goal here is to find new graphs that admits PGST. However it is more desirable to find PGST in graphs with large diameters. In [10], Godsil et al. established that the path on n vertices, denoted by P_n, exhibits PGST between the end vertices if and only if $n+1$ equals to either 2^m or p or $2p$, where p is an odd prime. Further investigations have been done in [5] on the existence of PGST in internal vertices of some paths P_n. In [20], van Bommel established a complete characterization of PGST in P_n between any pair of vertices. A complete characterization of cycles exhibiting PGST appears in [15]. There we see that a cycle on n vertices exhibits PGST if and only if n is a power of 2. In [7], Fan and Godsil showed that a double star $S_{k,k}$ admits PGST if and only if $4k+1$ is not a perfect square. A NEPS is a graph product which generalizes several well known graph products, namely, Cartesian product, Kronecker product etc. In [16], we see some NEPS of the path on three vertices having PGST. Also, in [1,2], PGST has been studied on corona products. Some other relevant results regarding state transfer can be found in [9,13,17,14,18]. In the present article, we find a class of circulant graphs admitting PGST. Also we find some

E-mail address: hiranmoy@iiitbh.ac.in.

https://doi.org/10.1016/j.disc.2017.12.012

circulant graphs not exhibiting pretty good state transfer. The results established in this article generalize several results appearing in [15] on circulant graphs admitting PGST.

Let $(\Gamma, +)$ be a finite abelian group and consider $S \setminus \{0\} \subseteq \Gamma$ with $\{-s : s \in S\} = S$. A Cayley graph over Γ with the symmetric set S is denoted by $Cay(\Gamma, S)$. The graph has the vertex set Γ where two vertices $a, b \in \Gamma$ are adjacent if and only if $a - b \in S$. The set S is called the connection set of $Cay(\Gamma, S)$. Let $\mathbb{Z}_n$ be the cyclic group of order n. A circulant graph is a Cayley graph over $\mathbb{Z}_n$. The cycle C_n, in particular, is the circulant graph over $\mathbb{Z}_n$ with the connection set $\{1, n-1\}$. The eigenvalues and its corresponding eigenvectors of a cycle are very well known. The eigenvalues of C_n are

$$\lambda_l = 2\cos\left(\frac{2l\pi}{n}\right), \; 0 \le l \le n-1, \tag{1}$$

and the corresponding eigenvectors are $\mathbf{v}_l = \left[1, \omega_n^l, \ldots, \omega_n^{l(n-1)}\right]^T$, where $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$ is the primitive nth root of unity.

In general, the eigenvalues and eigenvectors of a Cayley graph over an abelian group are also known in terms of characters of the abelian group. In [11], it appears that the eigenvectors of a Cayley graph over an abelian group are independent of the connection set. Therefore the sets of eigenvectors of two Cayley graphs defined over an abelian group can be chosen to be equal. Hence we have the following result.

Proposition 1.1. *If S_1 and S_2 are symmetric subsets of an abelian group Γ then adjacency matrices of the Cayley graphs $Cay(\Gamma, S_1)$ and $Cay(\Gamma, S_2)$ commute.*

Let G and H be two simple graphs on the same vertex set V, and the respective edge sets $E(G)$ and $E(H)$. The edge union of G and H, denoted $G \cup H$, is defined on the vertex set V whose edge set is $E(G) \cup E(H)$. Using Proposition 1.1 and the exponential nature of the transition matrix, we obtain the following result which allows us to find the transition matrix of an edge union of two edge disjoint Cayley graphs.

Proposition 1.2 ([17]). *Let Γ be a finite abelian group and consider two disjoint and symmetric subsets $S, T \subset \Gamma$. Suppose the transition matrices of $Cay(\Gamma, S)$ and $Cay(\Gamma, T)$ are $H_S(t)$ and $H_T(t)$, respectively. Then $Cay(\Gamma, S \cup T)$ has the transition matrix $H_S(t)H_T(t)$.*

Let $n \in \mathbb{N}$ and d be a proper divisor of n. We denote

$$S_n(d) = \{x \in \mathbb{Z}_n : gcd(x, n) = d\},$$

and for any set D of proper divisors of n, we define

$$S_n(D) = \bigcup_{d \in D} S_n(d).$$

The set $S_n(D)$ is called a gcd-set of $\mathbb{Z}_n$. A graph is called integral if all its eigenvalues are integers. The following theorem determines which circulant graphs are integral.

Theorem 1.3 ([19]). *A circulant graph $Cay(\mathbb{Z}_n, S)$ is integral if and only if S is a gcd-set.*

It is therefore clear that if a circulant graph $Cay(\mathbb{Z}_n, S)$ is non-integral then there exists a proper divisor d of n such that $S \cap S_n(d)$ is a non-empty proper subset of $S_n(d)$.

A graph G is said to exhibit perfect state transfer (PST) between a pair of vertices u and v if there exists $t \in \mathbb{R}$ such that $H(t)\mathbf{e}_u = \gamma \mathbf{e}_v$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$. The graph G is said to be periodic at a vertex u if there is $t (\neq 0) \in \mathbb{R}$ such that $H(t)\mathbf{e}_u = \gamma \mathbf{e}_u$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$. Moreover, the graph G is said to be periodic if it is periodic at all vertices at the same time. More information regarding periodicity and PST can be found in [8]. It can be observed that PGST is a generalization to PST. However, in [12], we observe in Proposition 1.3.1 that if a graph is periodic then it admits PST if and only if it admits PGST. It is worth mentioning the fact presented in [18] that a circulant graph is periodic if and only if the graph is integral. In [3], Bašić considered integral circulant graphs and presented a complete characterization of integral circulant networks exhibiting PST. This motivates us to study non-integral circulant graphs for PGST.

2. Pretty good state transfer on circulant graphs

Let us recall some important observations on circulant graphs exhibiting PGST (see [15]). We include the following discussion for convenience. Suppose G is a graph with adjacency matrix A. It is well known that the matrix P of an automorphism of G commutes with A. By the spectral decomposition of the transition matrix $H(t)$ of G, we conclude that $H(t)$ is a polynomial in A. Therefore P commutes with $H(t)$ as well. Suppose G admits PGST between two vertices u and v. Then there exists a sequence $\{t_k\}$ of real numbers and $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that

$$\lim_{k \to \infty} H(t_k)\mathbf{e}_u = \gamma \mathbf{e}_v.$$