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List 3-dynamic coloring of graphs with small maximum average degree

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ABSTRACT

An r-dynamic k-coloring of a graph G is a proper k-coloring such that for any vertex v, there are at least $\min\{r, \deg_G(v)\}$ distinct colors in $N_G(v)$. The *r-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G. The *list r-dynamic chromatic number* of a graph G is denoted by $ch_r^d(G)$.

Recently, Loeb et al. (0000) showed that the list 3-dynamic chromatic number of a planar graph is at most 10. And Cheng et al. (0000) studied the maximum average condition to have $\chi_3^d(G) \le 4,\ 5$, or 6. On the other hand, Song et al. (2016) showed that if G is planar with girth at least 6, then $\chi_r^d(G) \le r+5$ for any $r \ge 3$.

In this paper, we study list 3-dynamic coloring in terms of maximum average degree. We show that $ch_3^d(G) \le 6$ if $mad(G) < \frac{18}{7}$, $ch_3^d(G) \le 7$ if $mad(G) < \frac{14}{5}$, and $ch_3^d(G) \le 8$ if $mad(G) < 3$. All of the bounds are tight.

1. Introduction

Let k be a positive integer. A proper k-coloring $\phi : V(G) \to \{1, 2, \ldots, k\}$ of a graph G is an assignment of colors to the vertices of G so that any two adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the least k such that there exists a proper k-coloring of G. An r-dynamic k-coloring of a graph G is a proper k-coloring ϕ such that for each vertex $v \in V(G)$, either the number of distinct colors in its neighborhood is at least r or the colors in its neighborhood are all distinct, that is, $|\phi(N_G(v))| = \min\{r, \deg_G(v)\}$. The *$r$-dynamic chromatic number* $\chi_r^d(G)$ of a graph G is the least k such that there exists an r-dynamic k-coloring of G.

A *list assignment* on a graph G is a function L that assigns each vertex v a set $L(v)$ which is a list of available colors at v. For a list assignment L of a graph G, we say G is *L-colorable* if there exists a proper coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is said to be *k-choosable* if for any list assignment L such that $|L(v)| \ge k$ for every vertex v, G is L-colorable.

For a list assignment L of G, we say that G is *r-dynamically L-colorable* if there exists an r-dynamic coloring ϕ such that $\phi(v) \in L(v)$ for every $v \in V(G)$. A graph G is *r-dynamically k-choosable* if for any list assignment L with $|L(v)| \ge k$ for every vertex v, G is r-dynamically L-colorable. The *list r-dynamic chromatic number* $ch_r^d(G)$ of a graph G is the least k such that G is r-dynamically k-choosable.

The notion of r-dynamic coloring was firstly introduced in [12], and then it was widely studied in [1,4,6–10]. Note that it was also studied in [2,13,14] with the name of *r-hued coloring*. Similar to the Wegner's conjecture [15], a conjecture about dynamic coloring of planar graphs was proposed in [13].

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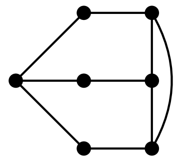

Fig. 1. A tight example for Theorem 1.3, $mad(H) = \frac{18}{7}$ and $ch_3^d(H) = 7$.

Conjecture 1.1. *Let G be a planar graph. Then*

$$\chi_r^d(G) \le \begin{cases} r+3 & \text{if } 1 \le r \le 2 \\ r+5 & \text{if } 3 \le r \le 7 \\ \lfloor \frac{3r}{2} \rfloor + 1 & \text{if } r \ge 8. \end{cases}$$

Song, Lai, and Wu [14] show that Conjecture 1.1 is true for planar graphs with girth at least 6.

Theorem 1.2 (*[14]*)**.** *If G is a planar graph with girth at least 6, $\chi_r^d(G) \le r+5$ for any $r \ge 3$.*

Recently, 3-dynamic coloring has been concerned. Loeb, Mahoney, Reiniger, and Wise [11] showed that $ch_3^d(G) \le 10$ if G is a planar graph. On the other hand, list 3-dynamic coloring was studied in [2] in terms of maximum average degree, where the *maximum average degree* of a graph G, $mad(G)$, is the maximum among the average degrees of the subgraphs of G. It was showed in [2] that $\chi_3^d(G) \le 6$ if $mad(G) < \frac{12}{5}$, $\chi_3^d(G) \le 5$ if $mad(G) < \frac{7}{3}$, and $\chi_3^d(G) \le 4$ if G has no C_5-component and $mad(G) < \frac{8}{3}$.

In this paper, we study list 3-dynamic coloring with maximum average degree condition. For each $k \in \{6, 7, 8\}$, we study the optimal value of maximum average degree to be $ch_3^d(G) \le k$. First, we give an optimal value of $mad(G)$ to be $ch_3^d(G) \le 6$, which improves a result in [2].

Theorem 1.3. *If $mad(G) < \frac{18}{7}$, then $ch_3^d(G) \le 6$.*

The bound on $mad(G)$ in Theorem 1.3 is tight. The graph H in Fig. 1 is a subcubic graph and so $ch_3^d(H) = ch(H^2)$, where the *square* of H, denoted by H^2, is the graph obtained by adding to H the edges connecting two vertices having a common neighbor in H. Note that $mad(H) = \frac{18}{7}$ and H^2 is isomorphic to K_7. Hence we have $ch(H^2) = ch_3^d(H) = 7$, which implies that the bound on $mad(G)$ in Theorem 1.3 is tight.

From the graph H, one can find infinitely many tight examples for Theorem 1.3. Given a graph H' with $mad(H') \le \frac{18}{7}$ and $ch_3^d(H') \le 7$, let G be a graph obtained from the union of two graphs H and H' by connecting with internally disjoint paths of length at least five such that the end vertices in H of the paths have degree two. Fig. 2 shows a way to construct such graphs. Note that there are at most three such paths since H has three vertices of degree two.

Remark. For a given graph F with $mad(F) \le \frac{18}{7}$, let F' be a graph obtained by adding a path of length ℓ ($\ell \ge 5$) to F such that the two end vertices x and y are in F and the other internal vertices are not. We will show that $mad(F') \le \frac{18}{7}$.

For a graph G, let ρ_G be a function defined on the power set of $V(G)$ such that $\rho_G(A) = 9|A| - 7|E(G[A])|$ for any $A \subset V(G)$, where $|A|$ denotes the number of vertices in A and $|E(G[A])|$ denotes the number of edges in the subgraph induced by A. Note that $\rho_G(A) \ge 0$ for any $A \subset V(G)$ if and only if $mad(G) \le \frac{18}{7}$.

Take any subset $A' \subset V(F')$. If $\{x, y\} \subset A'$, then

$$\rho_{F'}(A') \ge \rho_{F'}(A' \cap V(F)) + 9(\ell-1) - 7(\ell) \ge \rho_F(A' \cap V(F)) + 2\ell - 9 > \rho_F(A' \cap V(F)) \ge 0,$$

since $\ell \ge 5$. If $\{x, y\} \not\subset A'$, then

$$\rho_{F'}(A') = \rho_{F'}(A' \cap V(F)) + 9|A' - V(F)| - 7|A' - V(F)| \ge \rho_F(A' \cap V(F)) \ge 0.$$

Therefore, $mad(F') \le \frac{18}{7}$.

Thus, from the above Remark, it follows that $mad(G) = mad(H) = \frac{18}{7}$. Since $\deg_G(v) = 3$ for any $v \in V(H)$ and the distance (in G) between two vertices in $V(H)$ is at most two, all seven vertices in $V(H)$ should get distinct colors in a 3-dynamic coloring of G and so $ch_3^d(G) = \chi_3^d(G) = 7$.

We also study the value of $mad(G)$ to be $ch_3^d(G) \le 7$.

Theorem 1.4. *If $mad(G) < \frac{14}{5}$, then $ch_3^d(G) \le 7$.*

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