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Maximizing the number of x-colorings of 4-chromatic graphs

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ABSTRACT

Let $\mathcal{C}_4(n)$ be the family of all connected 4-chromatic graphs of order n. Given an integer $x \geq 4$, we consider the problem of finding the maximum number of x-colorings of a graph in $\mathcal{C}_4(n)$. It was conjectured that the maximum number of x-colorings is equal to $(x)_{\downarrow 4}(x-1)^{n-4}$ and the extremal graphs are those which have clique number 4 and size $n+2$.

In this article, we reduce this problem to a *finite* family of graphs. We show that there exists a finite family $\mathcal{F}$ of connected 4-chromatic graphs such that if the number of x-colorings of every graph G in $\mathcal{F}$ is less than $(x)_{\downarrow 4}(x-1)^{|V(G)|-4}$ then the conjecture holds to be true.

1. Introduction

In recent years problems of maximizing the number of colorings over various families of graphs have received a considerable amount of attention in the literature, see, for example, [1,4–7,9,10,12,16]. A natural graph family to look at is the family of connected graphs with fixed chromatic number and fixed order. Let $\mathcal{C}_k(n)$ be the family of all connected k-chromatic graphs of order n. What is the maximum number of k-colorings among all graphs in $\mathcal{C}_k(n)$? Or more generally, for an integer $x \geq k$, what is the maximum number of x-colorings of a graph in $\mathcal{C}_k(n)$ and what are the extremal graphs? The answer to this question depends on the chromatic number k. When $k \leq 3$, the answer to this question is known and when $k \geq 4$ the problem is wide open. It is well known that for $k=2$ and $x \geq 2$, the maximum number of x-colorings of a graph in $\mathcal{C}_2(n)$ is equal to $x(x-1)^{n-1}$, and the extremal graphs are trees when $x \geq 3$ [4]. For $k=3$, Tomescu [14] settled the problem by showing the following:

Theorem 1.1 ([14, Corollary 1]). *If G is a graph in $\mathcal{C}_3(n)$ then*

$$\pi(G,x) \leq (x-1)^n - (x-1) \quad \textit{for odd } n$$

and

$$\pi(G,x) \leq (x-1)^n - (x-1)^2 \quad \textit{for even } n$$

for every integer $x \geq 3$. Furthermore, the unique extremal graph is the odd cycle C_n when n is odd and odd cycle with a vertex of degree 1 attached to the cycle (denoted C^1_{n-1}) when n is even.

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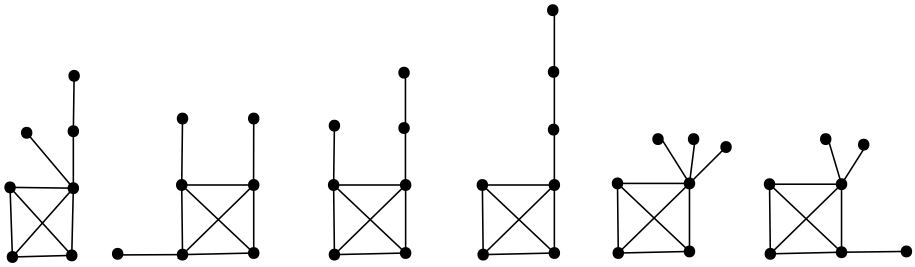

Fig. 1. The graphs in the family $\mathcal{C}_4^*(7)$.

Let $\mathcal{C}_k^*(n)$ be the set of all graphs in $\mathcal{C}_k(n)$ which have clique number k and size $\binom{k}{2}+n-k$ (see Fig. 1). It is easy to see that if $G \in \mathcal{C}_k^*(n)$ then $\pi(G,x) = (x)_{\downarrow k}(x-1)^{n-k}$ where $(x)_{\downarrow k}$ is the kth falling factorial $x(x-1)(x-2)\cdots(x-k+1)$. Tomescu [13] conjectured that when $k \geq 4$, the maximum number of k-colorings of a graph in $\mathcal{C}_k(n)$ is equal to $k!(k-1)^{n-k}$ and extremal graphs belong to $\mathcal{C}_k^*(n)$.

Conjecture 1.2 (*[13]*)**.** *If $G \in \mathcal{C}_k(n)$ where $k \geq 4$ then*

$$\pi(G,k) \leq k!\,(k-1)^{n-k}$$

and extremal graphs belong to $\mathcal{C}_k^*(n)$.

The conjecture above was later extended to all x-colorings with $x \geq 4$.

Conjecture 1.3 (*[4, p. 315]*)**.** *Let G be a graph in $\mathcal{C}_k(n)$ where $k \geq 4$. Then for every $x \in \mathbb{N}$ with $x \geq k$*

$$\pi(G,x) \leq (x)_{\downarrow k}(x-1)^{n-k}.$$

Moreover, the equality holds if and only if G belongs to $\mathcal{C}_k^*(n)$.

Several authors have studied Conjecture 1.3. In [14], Conjecture 1.3 was proven for $k = 4$ under the additional condition that graphs are planar:

Theorem 1.4 (*[14, Theorem 3]*)**.** *If G is a planar graph in $\mathcal{C}_4(n)$ then*

$$\pi(G,x) \leq (x)_{\downarrow 4}(x-1)^{n-4}$$

for every integer $x \geq 4$ and furthermore equality holds if and only if G belongs to $\mathcal{C}_4^*(n)$.

Also, in [1, Theorem 2.5] Conjecture 1.3 was proven for every $k \geq 4$, provided that $x \geq n-2+\left(\binom{n}{2}-\binom{k}{2}-n+k\right)^2$, and in [7, Theorem 2.1] it was proven for every $k \geq 4$ under the additional condition that the independence number of the graph is at most 2. In this article, our main result is Theorem 1.5 which reduces Conjecture 1.3 (for $k = 4$) to a *finite* family of 4-chromatic graphs.

Theorem 1.5. *There exists a finite family $\mathcal{F}$ of 3-connected nonplanar 4-chromatic graphs such that if every graph G in $\mathcal{F}$ satisfies $\pi(G,x) < (x)_{\downarrow 4}(x-1)^{|V(G)|-4}$ for all $x \in \mathbb{N}$ with $x \geq 4$, then Conjecture 1.3 holds to be true.*

Note that our result does not say anything about the number of graphs in this finite family. Our proof relies on Theorem 3.4, so it might be helpful to analyze its proof in order to determine how big this finite family is.

2. Terminology and background

Let $V(G)$ and $E(G)$ be the vertex set and edge set of a (finite, undirected) graph G, respectively. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. For a nonnegative integer x, a (proper) *x-coloring* of G is a function $f : V(G) \to \{1,\ldots,x\}$ such that $f(u) \neq f(v)$ for every $uv \in E(G)$. The *chromatic number* $\chi(G)$ is smallest x for which G has an x-coloring and G is called *k-chromatic* if $\chi(G) = k$. Let $\pi(G,x)$ denote the *chromatic polynomial of* G. For nonnegative integers x, the polynomial $\pi(G,x)$ counts the number of x-colorings of G.

Let $G+e$ be the graph obtained from G by adding an edge e and G/e be the graph formed from G by *contracting* edge e (or non-edge e if $e \notin E(G)$). For $e \notin E(G)$, observe that

$$\chi(G) = \min\{\chi(G+e),\ \chi(G/e)\}$$

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