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Pentavalent symmetric graphs admitting transitive non-abelian characteristically simple groups

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ABSTRACT

Let Γ be a graph and let G be a group of automorphisms of Γ. The graph Γ is called *G-normal* if G is normal in the automorphism group of Γ. Let T be a finite non-abelian simple group and let $G = T^l$ with $l \geq 1$. In this paper we prove that if every connected pentavalent symmetric T-vertex-transitive graph is T-normal, then every connected pentavalent symmetric G-vertex-transitive graph is G-normal. This result, among others, implies that every connected pentavalent symmetric G-vertex-transitive graph is G-normal except T is one of 57 simple groups. Furthermore, every connected pentavalent symmetric G-regular graph is G-normal except T is one of 20 simple groups, and every connected pentavalent G-symmetric graph is G-normal except T is one of 17 simple groups.

1. Introduction

Throughout this paper, all groups and graphs are finite, and all graphs are simple and undirected. Denote by $\mathbb{Z}_n$, D_n, A_n and S_n the cyclic group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and *regular* if it is semiregular and transitive. For a graph Γ, we denote its vertex set and automorphism group by $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$, respectively. The graph Γ is said to be *G-vertex-transitive* or *G-regular* for $G \leq \mathrm{Aut}(\Gamma)$ if G acts transitively or regularly on $V(\Gamma)$ respectively, and *G-symmetric* if G acts transitively on the arc set of Γ (an arc is an ordered pair of adjacent vertices). In particular, Γ is *vertex-transitive* or *symmetric* if it is $\mathrm{Aut}(\Gamma)$-vertex-transitive or $\mathrm{Aut}(\Gamma)$-symmetric, respectively. A graph Γ is said to be *G-normal* for $G \leq \mathrm{Aut}(\Gamma)$ if G is normal in $\mathrm{Aut}(\Gamma)$.

For a non-abelian simple group T, T-vertex-transitive graphs have received wide attentions, specially for the two extreme cases: T-symmetric graphs and T-regular graphs. It was shown in [2] that a connected pentavalent symmetric T-vertex-transitive graph Γ is either T-normal or $\mathrm{Aut}(\Gamma)$ contains a non-abelian simple normal subgroup L such that $T \leq L$ and (T, L) is one of 58 possible pairs of non-abelian simple groups.

A T-regular graph is also called a *Cayley graph* over T, and the Cayley graph is called *normal* if it is T-normal. Investigation of Cayley graphs over a non-abelian simple group is currently a hot topic in algebraic graph theory. One of the most remarkable achievements is the complete classification of connected trivalent symmetric non-normal Cayley graphs over non-abelian simple groups. This work was began in 1996 by Li [12], and he proved that a connected trivalent symmetric Cayley graph Γ over a non-abelian simple group T is either normal or $T = A_5, A_7, \mathrm{PSL}(2, 11), M_{11}, A_{11}, A_{15}, M_{23}, A_{23}$ or A_{47}. In 2005, Xu et al. [16] proved that either Γ is normal or $T = A_{47}$, and two years later, Xu et al. [17] further showed that if $T = A_{47}$ and Γ is not normal, then Γ must be 5-arc-transitive and up to isomorphism there are exactly two such graphs. Du et al. [2]

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showed that a connected pentavalent symmetric Cayley graph Γ over T is either normal, or Aut(Γ) contains a non-abelian simple normal subgroup L such that $T \leq L$ and (T, L) is one of 13 possible pairs of non-abelian simple groups.

For T-symmetric graphs, Fang and Praeger [4,5] classified such graphs when T is a Suzuki or Ree simple group acting transitively on the set of 2-arcs of the graphs. For a connected cubic T-symmetric graph Γ, it was proved by Li [12] that either Γ is T-normal or $(T, \mathrm{Aut}(\Gamma)) = (A_7, A_8), (A_7, S_8), (A_7, 2.A_8), (A_{15}, A_{16})$ or $(\mathrm{GL}(4, 2), \mathrm{AGL}(4, 2))$. Fang et al. [3] proved that none of the above five pairs can happen, that is, T is always normal in Aut(Γ). Du et al. [2] showed that a connected pentavalent T-symmetric graph Γ is either T-normal or Aut(Γ) contains a non-abelian simple normal subgroup L such that $T \leq L$ and (T, L) is one of 17 possible pairs of non-abelian simple groups.

Let G be the characteristically simple group T^l with $l \geq 1$. In this paper, we extend the above results on connected pentavalent T-vertex-transitive graphs to G-vertex-transitive graphs.

Theorem 1.1. *Let T be a non-abelian simple group and let $G = T^l$ with $l \geq 1$. Assume that every connected pentavalent symmetric T-vertex-transitive graph is T-normal. Then every connected pentavalent symmetric G-vertex-transitive graph is G-normal.*

In 2011, Hua et al. [10] proved that if every connected cubic symmetric T-vertex-transitive graph is T-normal, then every connected cubic symmetric G-vertex-transitive graph is G-normal. By Theorem 1.1 and [2, Theorem 1.1], we have the following corollaries.

Corollary 1.2. *Let T be a non-abelian simple group and let $G = T^l$ with $l \geq 1$. Then every connected pentavalent symmetric G-vertex-transitive graph is G-normal except for $T = PSL(2, 8)$, $\Omega_8^-(2)$ or A_{n-1} with $n \geq 6$ and $n \big| 2^9 \cdot 3^2 \cdot 5$.*

Corollary 1.3. *Let T be a non-abelian simple group and let $G = T^l$ with $l \geq 1$. Then every connected pentavalent G-symmetric graph is G-normal except for $T = A_{n-1}$ with $n = 2 \cdot 3, 2^2 \cdot 3, 2^4, 2^3 \cdot 3, 2^5, 2^2 \cdot 3^2, 2^4 \cdot 3, 2^3 \cdot 3^2, 2^5 \cdot 3, 2^4 \cdot 3^2, 2^6 \cdot 3, 2^5 \cdot 3^2, 2^7 \cdot 3, 2^6 \cdot 3^2, 2^7 \cdot 3^2, 2^8 \cdot 3^2$ or $2^9 \cdot 3^2$.*

Corollary 1.4. *Let T be a non-abelian simple group and let $G = T^l$ with $l \geq 1$. Then every connected pentavalent symmetric G-regular graph is G-normal except for $T = PSL(2, 8)$, $\Omega_8^-(2)$ or A_{n-1} with $n = 2 \cdot 3, 2^3, 3^2, 2 \cdot 5, 2^2 \cdot 3, 2^2 \cdot 5, 2^3 \cdot 3, 2^3 \cdot 5, 2^2 \cdot 3 \cdot 5, 2^4 \cdot 5, 2^3 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5, 2^6 \cdot 3 \cdot 5, 2^5 \cdot 3^2 \cdot 5, 2^7 \cdot 3 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^7 \cdot 3^2 \cdot 5$ or $2^9 \cdot 3^2 \cdot 5$.*

2. Preliminaries

In this section, we describe some preliminary results which will be used later. The first one is the vertex stabilizers of connected pentavalent symmetric graphs. By [7, Theorem 1.1], we have the following proposition.

Proposition 2.1. *Let Γ be a connected pentavalent G-symmetric graph with $v \in V(\Gamma)$. Then $G_v \cong \mathbb{Z}_5, D_5, D_{10}, F_{20}, F_{20} \times \mathbb{Z}_2, F_{20} \times \mathbb{Z}_4, A_5, S_5, A_4 \times A_5, S_4 \times S_5, (A_4 \times A_5) \rtimes \mathbb{Z}_2$, $\mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $A\Sigma L(2, 4)$, $A\Gamma L(2, 4)$ or $\mathbb{Z}_2^6 \rtimes \Gamma L(2, 4)$, where F_{20} is the Frobenius group of order 20, $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$. In particular, $|G_v| = 5, 2 \cdot 5, 2^2 \cdot 5, 2^2 \cdot 5, 2^3 \cdot 5, 2^4 \cdot 5, 2^2 \cdot 3 \cdot 5, 2^3 \cdot 3 \cdot 5, 2^4 \cdot 3^2 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^5 \cdot 3^2 \cdot 5, 2^6 \cdot 3 \cdot 5, 2^6 \cdot 3^2 \cdot 5, 2^7 \cdot 3 \cdot 5, 2^7 \cdot 3^2 \cdot 5$ or $2^9 \cdot 3^2 \cdot 5 \cdot 5$, respectively.*

Connected pentavalent symmetric graphs admitting vertex-transitive non-abelian simple groups were classified in [2].

Proposition 2.2 (*[2, Theorem 1.1]*)**.** *Let T be a non-abelian simple group and Γ a connected pentavalent symmetric T-vertex-transitive graph. Then either $T \trianglelefteq Aut(\Gamma)$, or $T = \Omega_8^-(2)$, $PSL(2, 8)$ or A_{n-1} with $n \geq 6$ and $n \big| 2^9 \cdot 3^2 \cdot 5$.*

The following is straightforward (also see the short proof of [2, Lemma 3.2]).

Proposition 2.3. *Let Γ be a connected pentavalent symmetric G-vertex-transitive graph with $v \in V(\Gamma)$ and let $A = Aut(\Gamma)$. If $H \leq A$ and $GH \leq A$, then $|H|/|H \cap G| = |(GH)_v|/|G_v| \big| 2^9 \cdot 3^2 \cdot 5$, and if Γ is further G-symmetric then $|H|/|H \cap G| \big| 2^9 \cdot 3^2$.*

The following proposition follows the classification of three-factor simple groups.

Proposition 2.4 (*[11, Theorem I]*)**.** *Let G be a non-abelian simple $\{2, 3, 5\}$-group. Then $G = A_5, A_6$ or $\mathrm{PSU}(4, 2)$.*

By Guralnick [8, Theorem 1], we have the following proposition.

Proposition 2.5. *Let G be a non-abelian simple group with a subgroup H such that $|G : H| = p^a$ with p a prime and $a \geq 1$. Then*

(1) *$G = A_n$ and $H = A_{n-1}$ with $n = p^a$;*
(2) *$G = PSL(2, 11)$ and $H = A_5$ with $|G : H| = 11$;*
(3) *$G = M_{23}$ and $H = M_{22}$ with $|G : H| = 23$, or $G = M_{11}$ and $H = M_{10}$ with $|G : H| = 11$;*
(4) *$G = PSU(4, 2) \cong PSp(4, 3)$ and H is the parabolic subgroup of index 27;*
(5) *$G = PSL(n, q)$ and H is the stabilizer of a line or hyperplane with $|G : H| = (q^n - 1)/(q - 1) = p^a$.*

By [13, Theorem 1] and Proposition 2.5, we have the following proposition.