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On the genus distributions of wheels and of related graphs

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ABSTRACT

We are concerned with families of graphs in which there is a single root-vertex of unbounded valence, and in which, however, there is a uniform upper bound for the valences of all the other vertices. Using a result of Zagier, we obtain formulas and recursions for the genus distributions of several such families, including the wheel graphs. We show that the region distribution of a wheel graph is approximately proportional to the sequence of Stirling numbers of the first kind. Stahl has previously obtained such a result for embeddings in surfaces whose genus is relatively near to the maximum genus. Here, we generalize Stahl's result to the entire genus distributions of wheels. Moreover, we derive the genus distributions for four other graph families that have some similarities to wheels.

1. Introduction

An ***embedding*** of a graph G into an orientable surface S_k is a ***cellular embedding***, i.e., the interior of every face is homeomorphic to an open disc. Two embeddings of G are ***equivalent*** if they have the same rotation system, which happens if and only if there is an ambient isotopy from one embedding to the other. By the ***number of embeddings***, we mean the number of equivalence classes under this relation. We denote the number of embeddings of G on the surface S_k (of genus k) by $g_G(k)$. The ***genus distribution*** of the graph G is the sequence

$$g_G(0),\ g_G(1),\ g_G(2),\ \dots .$$

For any graph G, the ***region number*** $r_G(k)$ is the number of orientable embeddings of G such that the number of regions is k. The ***region distribution*** of the graph G is defined as the sequence

$$r_G(1),\ r_G(2),\ r_G(3),\ \dots .$$

If G is a connected graph with p vertices and q edges then the genus distribution and the region distribution are related as follows,

$$g_G(k) = r_G(2 - 2k - p + q),$$

according to Euler's polyhedral formula.

> Accordingly, any formula or recursion for the region distribution of a graph may also be regarded here as a formula or recursion of the genus distribution, and vice versa.

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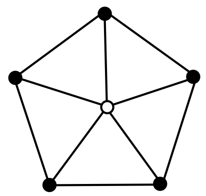

Fig. 1.1. The wheel graph W_5.

The ***minimum genus*** of a graph G, denoted $\gamma_{min}(G)$, and the ***maximum genus*** of G, denoted $\gamma_{max}(G)$, are the minimum value and maximum value of g, respectively, for which the graph G has an embedding into a surface of genus g. Duke's interpolation theorem asserts that for any integer k such that $\gamma_{min}(G) \le k \le \gamma_{max}(G)$, there exists an embedding of G into the surface S_k. As a source on topological graph theory, we recommend [6].

1.1. Embeddings of wheels

The **n-*wheel*** W_n is a graph formed by connecting a single vertex (at the hub) to each of the vertices of a n-cycle (along the rim). We regard the n-valent vertex as a root vertex, and we observe that all the vertices on the rim are 3-valent. Fig. 1.1 shows an embedding of the wheel graph W_5 in the sphere S_0.

We observe that there are six regions. We have $r_{W_5}(6) = 2$, since there are only two orientable embeddings of the wheel W_5 with six regions, the one shown in Fig. 1.1 and its mirror image. In general, it follows from the Euler polyhedral equation that the possible region numbers of a given graph all have the same parity as the number n of vertices on the rim of the wheel W_n.

The ***Stirling numbers of the first kind*** are defined to be the coefficients of the product

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Following [3], we use the Karamata notation $\left[{n \atop k}\right]$ for those Stirling numbers. It is well-known that the Stirling number $\left[{n \atop k}\right]$ counts the number of permutations in Σ_n that have exactly k orbits.

Stahl [12] observed that the characters of the symmetric group contain rich information regarding genus distributions of some kinds of graphs. He used character-theoretic methods in [14] to obtain the following result for the region distribution $r_{W_n}(k)$ of the n-wheel graph. We see that it is restricted to the relatively smaller possible numbers of regions (and, thus, to embeddings in surfaces whose genus is relatively close to maximum genus).

Theorem 1.1 (*[14]*)**.** *For* $1 \le k \le n^{\frac{1}{5}}$, *we have*

$$r_{W_n}(k) = \begin{cases} 0, & \text{if } n \equiv k \,(\text{mod } 2) \\ \dfrac{2^{n+1}\left[{n \atop k}\right]}{n}\left(1 + \dfrac{O(k^2)}{n}\right), & \text{otherwise.} \end{cases} \tag{1.1}$$

Roughly speaking, we say that the values of $r_{W_n}(k)$ are approximately proportional to the unsigned Stirling numbers of the first kind.

1.2. An outline of this paper

In Section 2, we review and develop what we need from character theory. In Section 3, we use a formula of [16] to generalize Theorem 1.1. We show that Eq. (1.1) holds for all possible numbers k of regions of an embedding of W_n. In subsequent sections, we continue with character-theoretic methods and thereby obtain formulas and recursions for the genus distributions of three other families of graphs with the following property:

> The valences of the vertices are bounded, except for a single root-vertex, whose valence is unbounded.

The first family with this property whose genus distribution was derived was the sequence of bouquets B_n. A formula and a recursion for the genus distribution of the bouquets was obtained by [5], based on a formula of [7]. Generalized fan graphs were next [1]. The derivations of the formulas and recursions for both bouquets and fans depend on the theory of characters of the symmetric group.