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Maximum weight stable set in (P_7, bull)-free graphs and ($S_{1,2,3}$, bull)-free graphs

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ABSTRACT

We give a polynomial time algorithm that finds the maximum weight stable set in a graph that does not contain an induced path on seven vertices or a bull (the graph with vertices a, b, c, d, e and edges ab, bc, cd, be, ce). With the same arguments we also give a polynomial algorithm for any graph that does not contain $S_{1,2,3}$ or a bull.

1. Introduction

In a graph G, a *stable set* (or *independent set*) is a subset of pairwise non-adjacent vertices. The *Maximum Stable Set problem* (shortened as MSS) is the problem of finding a stable set of maximum cardinality. In the weighted version, let $w : V(G) \rightarrow \mathbb{N}$ be the weight function over the set of vertices. The weight of any subset of vertices is defined as the sum of the weight of all its elements. The *Maximum Weight Stable Set problem* (shortened as MWSS) is the problem of finding a stable set of maximum weight. It is known that MSS and MWSS are NP-hard in general [10].

Given a set of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$*-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{F}$. If $\mathcal{F}$ is composed of only one element F, we say that G is F-free. On the other hand, we say that G *contains* F when F is isomorphic to an induced subgraph of G. For any integer k, we let P_k, C_k and K_k denote respectively the chordless path on k vertices, the chordless cycle on k vertices, and the complete graph on k vertices. The *claw* is the graph with four vertices a, x, y, z and three edges ax, ay, az. Let $S_{i,j,k}$ be the graph obtained from a claw by subdividing its edges into respectively i, j and k edges. Let us say that a graph is *special* if every component of the graph is a path or an $S_{i,j,k}$ for any i, j, k.

- Alekseev [1] proved that MSS remains NP-hard in the class of $\mathcal{F}$-free graphs whenever $\mathcal{F}$ is a finite set of graphs such that no member of $\mathcal{F}$ is special.
- Several authors [9,18,23,24,27] proved that MWSS can be solved in polynomial time for claw-free graphs ($S_{1,1,1}$-free graphs).
- Lozin and Milanič [16] proved that MWSS can be solved in polynomial time for fork-free graphs ($S_{1,1,2}$-free graphs).
- Lokshtanov, Vatshelle and Villager [15] proved that MWSS can be solved in polynomial time for P_5-free graphs ($S_{0,2,2}$-free graphs).

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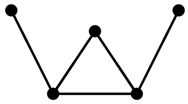

Fig. 1. The bull.

The results above settle the complexity of MWSS in F-free graph whenever F is a connected special graph on at most five vertices. Therefore the new frontier to explore is when the forbidden induced subgraph has six or more vertices. There are several results on the existence of a polynomial time algorithm for MWSS in subclasses of P_6-free graphs [12,14,17,19,21,22]. Mosca [20] proved that MWSS is solvable in polynomial for the class of (P_7, banner)-free graphs. Brandstädt and Mosca [2] proved that there exists a polynomial time algorithm for the MWSS problem in the class of (P_7, K_3)-free graphs. The *bull* is the graph with vertices a, b, c, d, e and edges ab, bc, cd, be, ce (see Fig. 1). Our main results are the following two theorems.

Theorem 1.1. *The Maximum Weight Stable Set problem can be solved in polynomial time in the class of (P_7, bull)-free graphs.*

Theorem 1.2. *The Maximum Weight Stable Set problem can be solved in polynomial time in the class of ($S_{1,2,3}$, bull)-free graphs.*

Theorem 1.1 generalizes the main results in [2] and [17], and Theorem 1.2 generalizes the main results in [13] and [17].

Our paper is organized as follows. In the rest of this section we recall some definitions, notations and well known results. In Section 2 we develop a structural description that we can use to solve the MWSS efficiently. In Section 3, thanks to the detailed structure, we show how to solve the MWSS in polynomial time in the class of (P_7, bull)-free graphs. In Section 4, we show how to solve the MWSS in polynomial time in the class of ($S_{1,2,3}$, bull)-free graphs.

Let G be a graph. For any vertex $v \in V(G)$, we denote by $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ the *neighborhood* of v. For any $S \subseteq V(G)$ we denote by $G[S]$ the *induced subgraph* of G with vertex-set S. For any $X \subseteq V(G)$, we may write $G \setminus X$ instead of $G[V(G) \setminus X]$. For any $S \subseteq V(G)$ and $x \in V(G)$, we let $N_S(x)$ stand for $N(x) \cap S$. For two sets $K, S \subseteq V(G)$, we say that K is *complete* to S if every vertex of K is adjacent to every vertex of S, and we say that K is *anticomplete* to S if no vertex of K is adjacent to any vertex of S. A *homogeneous set* is a set $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ is either complete to S or anticomplete to S. A homogeneous set is *proper* if it contains at least two vertices and is different from $V(G)$. A graph is *prime* if it has no proper homogeneous set.

A *hole* in a graph is any induced cycle on at least four vertices. An *antihole* is the complement of a hole. A graph G is perfect if, for every induced subgraph G' of G, the chromatic number of G' is equal to the maximum clique size in G'. The Strong Perfect Graph Theorem [5] establishes that a graph is perfect if and only if it contains no odd hole and no odd antihole.

In a series of papers [3,4] Chudnovsky established a decomposition theorem for all bull-free graphs. Based on this decomposition, Thomassé, Trotignon and Vušković [28] proved that the MWSS problem is fixed-parameter tractable in the class of bull-free graphs. It might be that these results could be adapted so as to yield an alternate proof of Theorems 1.1 and 1.2. However we are able to avoid using the rather complex machinery of [28] and [3,4]. Our proof is based on conceptually simple ideas derived from [2] and is self-contained.

2. Structural description

A class of graphs is *hereditary* if, for every graph G in the class, every induced subgraph of G is also in the class. For example, for any set $\mathcal{F}$ of graphs, the class of $\mathcal{F}$-free graphs is hereditary. We will use the following theorem of Lozin and Milanič [16].

Theorem 2.1 ([16]). *Let $\mathcal{G}$ be a hereditary class of graphs. Suppose that there is a constant $c \geq 1$ such that the MWSS problem can be solved in time $O(|V(G)|^c)$ for every prime graph G in $\mathcal{G}$. Then the MWSS problem can be solved in time $O(|V(G)|^c + |E(G)|)$ for every graph G in $\mathcal{G}$.*

The classes of (P_7, bull)-free graphs and ($S_{1,2,3}$, bull)-free graphs are hereditary. Hence, in order to prove Theorems 1.1 and 1.2 it suffices to prove them for prime graphs.

In a graph G, let H be a subgraph of G. For each $k > 0$, a *k-neighbor* of H is any vertex in $V(G) \setminus V(H)$ that has exactly k neighbors in H. The following two lemmas are straightforward and we omit their proof.

Lemma 2.2 ([17]). *Let G be a bull-free graph. Let C be an induced C_5 in G, with vertices $c_1, \ldots, c_5$ and edges c_ic_{i+1} for each i modulo 5. Then:*

- *Any 2-neighbor of C is adjacent to c_i and c_{i+2} for some i.*
- *Any 3-neighbor of C is adjacent to c_i, c_{i+1} and c_{i+2} for some i.*
- *If a non-neighbor of C is adjacent to a k-neighbor of C, then $k \in \{1, 2, 5\}$.*

Lemma 2.3. *Let G be a bull-free graph. Let C be an induced C_7 in G, with vertices $c_1, \ldots, c_7$ and edges c_ic_{i+1} for each i modulo 7. Then:*