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On hypergraphs without loose cycles

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ABSTRACT

Recently, Mubayi and Wang showed that for $r \geq 4$ and $\ell \geq 3$, the number of n-vertex r-graphs that do not contain any loose cycle of length ℓ is at most $2^{O(n^{r-1}(\log n)^{(r-3)/(r-2)})}$. We improve this bound to $2^{O(n^{r-1}\log\log n)}$.

1. Introduction

Let two graphs G and H be given. The graph G is called H-*free* if it does not contain any copy of H as a subgraph. One of the central problems in graph theory is to determine the extremal and typical properties of H-free graphs on n vertices. For example, one of the first influential results of this type is the Erdős–Kleitman–Rothschild theorem [4], which, for instance, implies that the number of triangle-free graphs with vertex set $[n] = \{1, \ldots, n\}$ is $2^{n^2/4+o(n^2)}$. This has inspired a great deal of work on counting the number of H-free graphs. For an overview of this line of research, the reader is referred to, e.g., [3,10]. For a recent, exciting result in the area, see [8], which also contains a good discussion of the general area, with several pointers to the literature. These problems are closely related to the so-called *Turán problem*, which asks to determine the maximum possible number of edges in an H-free graph. More precisely, given an r-uniform hypergraph (or r-graph) H, the *Turán number* $\mathrm{ex}_r(n, H)$ is the maximum number of edges in an r-graph G on n vertices that is H-free. Let $\mathrm{Forb}_r(n, H)$ be the set of all H-free r-graphs with vertex set $[n]$. Noting that the subgraphs of an H-free r-graph G are also H-free, we trivially see that $|\mathrm{Forb}_r(n, H)| \geqslant 2^{\mathrm{ex}_r(n,H)}$, by considering an H-free r-graph G on $[n]$ with the maximum number of edges and all its subgraphs. On the other hand for, fixed r and H,

$$|\mathrm{Forb}_r(n, H)| \leqslant \sum_{1\leqslant i\leqslant \mathrm{ex}_r(n,H)} \binom{\binom{n}{r}}{i} = 2^{O(\mathrm{ex}_r(n,H)\log n)}. \tag{1}$$

Hence the above simple bounds differ by a factor of $\log n$ in the exponent, and all existing results support that this $\log n$ factor should be unnecessary, i.e., the trivial lower bound should be closer to the truth.

There are very few results in the case $r > 2$ and $\mathrm{ex}_r(n, H) = o(n^r)$. The only known case is when H consists of two edges sharing t vertices [1,5]. Very recently, Mubayi and Wang [9] studied $|\mathrm{Forb}_r(n, H)|$ when H is a loose cycle. Given $\ell \geqslant 3$, an r-*uniform loose cycle* C_ℓ^r is an $\ell(r-1)$-vertex r-graph whose vertices can be ordered cyclically in such a way that the edges are sets of consecutive r vertices and every two consecutive edges share exactly one vertex. When r is clear from the context, we simply write C_ℓ.

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Theorem 1 (*[9]*). *For every $\ell \geqslant 3$ and $r \geqslant 4$, there exists $c = c(r, \ell)$ such that*

$$|Forb_r(n, C_\ell)| < 2^{cn^{r-1}(\log n)^{(r-3)/(r-2)}} \tag{2}$$

for all n. For $\ell \geqslant 4$ even, there exists $c = c(\ell)$ such that $|Forb_3(n, C_\ell)| < 2^{cn^2}$ for all n.

Since $\mathrm{ex}_r(n, C_\ell) = \Omega(n^{r-1})$ for all $r \geqslant 3$ [6,7], Theorem 1 implies that $|\mathrm{Forb}_3(n, C_\ell)| = 2^{\Theta(n^2)}$ for even $\ell \geqslant 4$. Mubayi and Wang also conjecture that similar results should hold for $r = 3$ and all $\ell \geqslant 3$ odd and for all $r \geqslant 4$ and $\ell \geqslant 3$, i.e., $|\mathrm{Forb}_r(n, C_\ell)| = 2^{\Theta(n^{r-1})}$ for all such r and ℓ. In this note we give the following improvement of Theorem 1 for $r \geqslant 4$.

Theorem 2. *For every $\ell \geqslant 3$ and $r \geqslant 4$, we have*

$$|Forb_r(n, C_\ell)| < 2^{2r^2 \ell n^{r-1} \log\log n} \tag{3}$$

for all sufficiently large n.

In what follows, logarithms have base 2.

2. Edge-colored *r*-graphs

Let $r \geqslant 2$ be an integer. An *r-uniform hypergraph* G (or *r-graph*) on a *vertex set* X is a collection of r-element subsets of X, called *hyperedges* or simply *edges*. The vertex set X of G is denoted $V(G)$. We write $e(G)$ for the number of edges in G. An *r-partite r-graph* G is an r-graph together with a vertex partition $V(G) = V_1 \cup \cdots \cup V_r$, such that every edge of G contains exactly one vertex from each V_i ($i \in [n]$). If all such edges are present in G, then we say that G is *complete*. We call an r-partite r-graph *balanced* if all parts in its vertex partition have the same size. Let $K_r(s)$ denote the complete r-partite r-graph with s vertices in each vertex class.

We now introduce some key definitions from [9], which are also essential for us. Given an $(r-1)$-graph G with $V(G) \subseteq [n]$, a *coloring function* for G is a function $\chi : G \to [n]$ such that $\chi(e) = z_e \in [n] \setminus e$ for every $e \in G$. We call z_e the *color* of e. The pair (G, χ) is an *edge-colored* $(r-1)$-graph.

Given G, each edge-coloring χ of G gives an r-graph $G^\chi = \{e \cup \{z_e\} : e \in G\}$, called the *extension of G by* χ. When there is only one coloring that has been defined, we write G^* for G^χ. Clearly any subgraph $G' \subseteq G$ also admits an extension by χ, namely, $(G')^\chi = \{e \cup \{z_e\} : e \in G'\} \subseteq G^*$. If $G' \subseteq G$ and $\chi\restriction_{G'}$ is one-to-one and $z_e \notin V(G')$ for all $e \in G'$, then G' is called *strongly rainbow colored*. We state the following simple remark explicitly for later reference.

Remark 3. A strongly rainbow colored copy of C_ℓ^{r-1} in G' gives rise to a copy of C_ℓ^r in G^*.

The following definition is crucial.

Definition 4 ($g_r(n, \ell)$, *[9]*). For $r \geqslant 4$ and $\ell \geqslant 3$, let $g_r(n, \ell)$ be the number of edge-colored $(r-1)$-graphs G with $V(G) \subseteq [n]$ such that the extension G^* is C_ℓ^r-free.

The function $g_r(n, \ell)$ above counts the number of pairs (G, χ) with $G^\chi \in \mathrm{Forb}_r(n, C_\ell^r)$. Mubayi and Wang [9] proved that $g_r(n, \ell)$ is non-negligible in comparison with $|\mathrm{Forb}_r(n, C_\ell)|$ and were thus able to deduce Theorem 1. The following estimate on $g_r(n, \ell)$ is proved in [9].

Lemma 5 (*[9], Lemma 8*). *For every $r \geqslant 4$ and $\ell \geqslant 3$ there is $c = c(r, \ell)$ such that for all large enough n we have* $\log g_r(n, \ell) \leqslant cn^{r-1}(\log n)^{(r-3)/(r-2)}$.

We improve Lemma 5 as follows.

Lemma 6. *For every $r \geqslant 4$ and $\ell \geqslant 3$ we have*

$$\log g_r(n, \ell) \leqslant 2rn^{r-1} \log\log n \tag{4}$$

for all large enough n.

Theorem 2 can be derived from Lemma 6 in the same way that Theorem 1 is derived from Lemma 5 in [9]. It thus remains to prove Lemma 6.

3. Proof of Lemma 6

To bound $g_r(n, \ell)$, we should consider all possible $(r-1)$-graphs G and their 'valid' edge-colorings. Let an $(r-1)$-graph G be fixed. The authors of [9] consider decompositions of G into balanced complete $(r-1)$-partite $(r-1)$-graphs G_i, and obtain good estimates on the number of edge-colorings of each G_i. In our proof of Lemma 6, we also decompose G into balanced $(r-1)$-partite $(r-1)$-graphs G_i, but with each G_i not necessarily complete. We get our improvement because certain quantitative aspects of our decomposition are better, and similar estimates can be shown for the number of edge-colorings of each G_i.