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Star 5-edge-colorings of subcubic multigraphs

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A B S T R A C T

The *star chromatic index* of a mulitigraph G, denoted $\chi'_s(G)$, is the minimum number of colors needed to properly color the edges of G such that no path or cycle of length four is bi-colored. A multigraph G is *star k-edge-colorable* if $\chi'_s(G) \leq k$. Dvořák et al. (2013) proved that every subcubic multigraph is star 7-edge-colorable, and conjectured that every subcubic multigraph should be star 6-edge-colorable. Kerdjoudj, Kostochka and Raspaud considered the list version of this problem for simple graphs and proved that every subcubic graph with maximum average degree less than 7/3 is star list-5-edge-colorable. It is known that a graph with maximum average degree 14/5 is not necessarily star 5-edge-colorable. In this paper, we prove that every subcubic multigraph with maximum average degree less than 12/5 is star 5-edge-colorable.

1. Introduction

All multigraphs in this paper are finite and loopless; and all graphs are finite and without loops or multiple edges. Given a multigraph G, let $c : E(G) \to [k]$ be a proper edge-coloring of G, where $k \geq 1$ is an integer and $[k] := \{1, 2, \ldots, k\}$. We say that c is a *star k-edge-coloring* of G if no path or cycle of length four in G is bi-colored under the coloring c; and G is *star k-edge-colorable* if G admits a star k-edge-coloring. The *star chromatic index* of G, denoted $\chi'_s(G)$, is the smallest integer k such that G is star k-edge-colorable. As pointed out in [6], the definition of star edge-coloring of a graph G is equivalent to the star vertex-coloring of its line graph $L(G)$. Star edge-coloring of a graph was initiated by Liu and Deng [10], motivated by the vertex version (see [1,4,5,8,11]). Given a multigraph G, we use $|G|$ to denote the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, and $\Delta(G)$ the maximum degree of G, respectively. We use K_n and P_n to denote the complete graph and the path on n vertices, respectively. A multigraph G is *subcubic* if all its vertices have degree less than or equal to three. The *maximum average degree* of a multigraph G, denoted $\mathrm{mad}(G)$, is defined as the maximum of $2e(H)/|H|$ taken over all the subgraphs H of G. The following upper bound is a result of Liu and Deng [10].

Theorem 1.1 ([10]). *For every graph G of maximum degree $\Delta \geq 7$, $\chi'_s(G) \leq \lceil 16(\Delta - 1)^{\frac{3}{2}} \rceil$.*

Theorem 1.2 is a result of Dvořák, Mohar and Šámal [6], which gives an upper and a lower bound for complete graphs.

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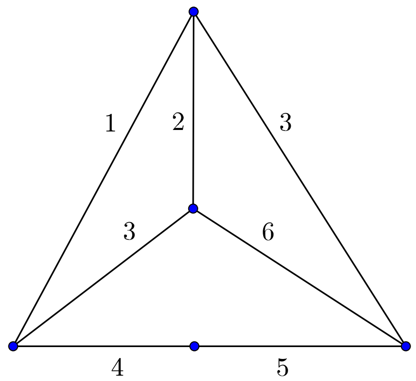

Fig. 1. A graph with maximum average degree 14/5 and star chromatic index 6.

Theorem 1.2 (*[6]*). *The star chromatic index of the complete graph K_n satisfies*

$$2n(1+o(1)) \le \chi_s'(K_n) \le n\,\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$, there exists a constant c such that $\chi_s'(K_n) \le cn^{1+\epsilon}$ for every integer $n \ge 1$.

The true order of magnitude of $\chi_s'(K_n)$ is still unknown. Applying the upper bound in Theorem 1.2 on $\chi_s'(K_n)$, an upper bound for $\chi_s'(G)$ of any graph G is also derived in [6].

Theorem 1.3 (*[6]*). *For every graph G of maximum degree Δ,*

$$\chi_s'(G) \le \chi_s'(K_{\Delta+1}) \cdot O\left(\frac{\log \Delta}{\log\log \Delta}\right)^2,$$

and so $\chi_s'(G) \le \Delta \cdot 2^{O(1)\sqrt{\log \Delta}}$.

It is worth noting that when Δ is large, Theorem 1.3 yields a near-linear upper bound for $\chi_s'(G)$, which greatly improves the upper bound obtained in Theorem 1.1. In the same paper, Dvořák, Mohar and Šámal [6] also considered the star chromatic index of subcubic multigraphs. To state their result, we need to introduce one notation. A graph G *covers* a graph H if there is a mapping $f : V(G) \to V(H)$ such that for any $uv \in E(G)$, $f(u)f(v) \in E(H)$, and for any $u \in V(G)$, f is a bijection between $N_G(u)$ and $N_H(f(u))$. They proved the following.

Theorem 1.4 (*[6]*). *Let G be a multigraph.*

(a) *If G is subcubic, then $\chi_s'(G) \le 7$.*
(b) *If G is cubic and has no multiple edges, then $\chi_s'(G) \ge 4$ and the equality holds if and only if G covers the graph of 3-cube.*

As observed in [6], $K_{3,3}$ is not star 5-edge-colorable but star 6-edge-colorable. No subcubic multigraphs with star chromatic index seven are known. Dvořák, Mohar and Šámal [6] proposed the following conjecture.

Conjecture 1.5 (*[6]*). *Let G be a subcubic multigraph. Then $\chi_s'(G) \le 6$.*

It was shown in [2] that every subcubic outerplanar graph is star 5-edge-colorable. Lei, Shi and Song [9] recently proved that every subcubic multigraph G with $\text{mad}(G) < 24/11$ is star 5-edge-colorable, and every subcubic multigraph G with $\text{mad}(G) < 5/2$ is star 6-edge-colorable. Kerdjoudj, Kostochka and Raspaud [7] considered the list version of star edge-colorings of simple graphs. They proved that every subcubic graph is star list-8-edge-colorable, and further proved the following stronger results.

Theorem 1.6 (*[7]*). *Let G be a subcubic graph.*

(a) *If $mad(G) < 7/3$, then G is star list-5-edge-colorable.*
(b) *If $mad(G) < 5/2$, then G is star list-6-edge-colorable.*

As mentioned above, $K_{3,3}$ has star chromatic index 6, and is bipartite and non-planar. The graph, depicted in Fig. 1, has star chromatic index 6, and is planar and non-bipartite. We see that not every bipartite, subcubic graph is star 5-edge-colorable; and not every planar, subcubic graph is star 5-edge-colorable. It remains unknown whether every bipartite, planar subcubic multigraph is star 5-edge-colorable. In this paper, we improve Theorem 1.6(a) by showing the following main result.