Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

An improved upper bound on the adjacent vertex distinguishing total chromatic number of graphs

Bojan Vučković

Matematički institut SANU, Kneza Mihaila 36, 11001 Beograd, p.p. 367, Serbia

ARTICLE INFO

Article history:
Received 28 December 2016
Received in revised form 6 October 2017
Accepted 11 October 2017
Available online xxxx

Keywords:
Adjacent vertex distinguishing total coloring
Maximum degree

ABSTRACT

An adjacent vertex distinguishing total k-coloring of a graph G is a proper total k-coloring of G such that any pair of adjacent vertices have different sets of colors. The minimum number k needed for such a total coloring of G is denoted by $\chi''_a(G)$. In this paper we prove that $\chi''_a(G) \leq 2\Delta(G) - 1$ if $\Delta(G) \geq 4$, and $\chi''_a(G) \leq \lceil \frac{5\Delta(G)+8}{3} \rceil$ in general. This improves a result in Huang et al. (2012) which states that $\chi''_a(G) \leq 2\Delta(G)$ for any graph with $\Delta(G) \geq 3$.

1. Introduction

A proper total k-coloring of a graph G is a mapping $\phi : V(G) \cup E(G) \to \{1, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. For a vertex $v \in V(G)$ and a proper total coloring ϕ, we define set $C_\phi(v)$ as $\{\phi(uv) | uv \in E(G)\} \cup \{\phi(v)\}$. The coloring ϕ is an *adjacent vertex distinguishing total coloring* or *avd-total coloring* if $C_\phi(v) \neq C_\phi(u)$ for every pair of adjacent vertices v and u. The *adjacent vertex distinguishing total chromatic number* $\chi''_a(G)$ of a graph G is the smallest integer k such that G has a k-avd total coloring.

For a graph G, having two adjacent vertices v and u with a degree $\Delta(G)$, both $C_\phi(v)$ and $C_\phi(u)$ have $\Delta(G) + 1$ elements. Since $C_\phi(v) \setminus C_\phi(u) \neq \emptyset$ is a necessary condition for these sets to be different, such a graph G has $\chi''_a(G)$ greater than or equal to $\Delta(G) + 2$. In fact, there exist many graphs with $\chi''_a(G) > \Delta(G) + 2$, for example, any complete graph of odd order has that property. An avd-total coloring was first introduced by Zhang et al. [7], where they proposed the following conjecture:

Conjecture 1.1. *For any graph G, $\chi''_a(G) \leq \Delta(G) + 3$.*

First we prove an upper bound on $\chi''_a(G)$ related to the chromatic number $\chi(G)$ and the maximum degree $\Delta(G)$.

Lemma 1.2. *For any graph G, $\chi''_a(G) \leq \chi(G) + \Delta(G)$.*

Proof. Let $k = \chi(G)$, $l = \Delta(G)$, $K = \{1, \ldots, k-1\}$ and $L = \{k, \ldots, k+l\}$. According to the famous Vizing's theorem [4], $\chi'(G) \leq \Delta(G) + 1$ for every graph G. This means that we can properly color the edges of G with colors from L. Let $V = V(G)$, and let $V_1, \ldots, V_k$ be color classes of a graph G. Since a proper coloring assigns different colors to every pair of adjacent vertices, each nonempty color class V_i, $1 \leq i \leq k$, is an independent set of vertices in G. For every $j \in K$, we color every vertex from V_j with color j. Since $K \cap L = \emptyset$, we have $\phi(v) \neq \phi(vu)$ for every $v \in (V \setminus V_k)$ and $u \in N(v)$. On the other hand, $i \in (C_\phi(v) \setminus C_\phi(u))$ for every $v \in V_i, u \in V_j$, $1 \leq i < j \leq k-1$, and thus $C_\phi(v) \neq C_\phi(u)$. We now color the remaining vertices, that is, vertices from V_k. Let v be a vertex from V_k. Since $\Delta(G) = l$, vertex v has at most l incident edges, and there is at least

E-mail address: b.vuckovic@turing.mi.sanu.ac.rs.

https://doi.org/10.1016/j.disc.2017.10.011

one color from L, say l_1, not used to color any of the edges incident with v. We color v with l_1. None of the vertices adjacent to v is colored from L, thus such a vertex coloring is proper. We know that $j \in C_\phi(u)$ for every $1 \le j \le k-1$ and every $u \in V_j$, and since $C_\phi(v) \cap K = \emptyset$ we have $C_\phi(v) \ne C_\phi(u)$. Therefore, the coloring described above is an avd-total coloring using $k + l = \chi(G) + \Delta(G)$ colors. □

Zhang et al. [7] showed the value of χ''_a for complete graphs:

Lemma 1.3.

$$\chi''_a(K_n) = \begin{cases} n+1, & \text{if } n \text{ is even,} \\ n+2, & \text{if } n \text{ is odd.} \end{cases}$$

Regarding an upper bound on the chromatic number of a graph, we know that $\chi(C_l) = 3$ for every odd l, and $\chi(K_n) = n+1$ for every n. Also, Brooks' theorem [1] states that $\chi(G) \le \Delta(G)$ for any graph different from an odd cycle and a complete graph. As a direct consequence of Lemma 1.2, Lemma 1.3 and Brooks' theorem we get a simpler proof of the following bound, given by Huang, Wang and Yan [2].

Corollary 1.4. *For any graph G with $\Delta(G) \ge 3$, we have $\chi''_a(G) \le 2\Delta(G)$.*

2. An improved upper bound

The following definition and lemma were given in somewhat different forms in [2].

Definition 2.1. Let G be a graph with $\chi(G) = k$, and let $V_1, \ldots, V_k$ be color classes of G. We say that $V_1, \ldots, V_k$ are dominant color classes if $N(v) \cap V_j \ne \emptyset$ for every $v \in V_i$, $1 < i \le k$, and every j, $1 \le j < i$. We call such a partitioning $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $V(G)$ a dominant partitioning.

Let $V_1, \ldots, V_k$ be color classes of a graph G. We can always obtain a dominant partitioning using the following simple algorithm.

Algorithm 1 Obtaining a dominant partitioning

Input: color classes $U_1, \ldots, U_k$
Output: dominant partitioning $\mathcal{P} = \{V_1, \ldots, V_k\}$

```
for all 1 ≤ i ≤ k do
    V_i ← ∅
end for
i ← 1
while i ≤ k do
    for all u ∈ U_i do
        Let j, j ≤ i, be the smallest integer for which N(u) ∩ V_j = ∅.
        Include u in V_j.
    end for
    i ← i + 1
end while
return {V_1, ..., V_k}
```

Lemma 2.2. *For any graph G with $\chi(G) = k$, there exists a dominant partitioning $\mathcal{P} = \{V_1, \ldots, V_k\}$ of $V(G)$.*

Proof. The previous algorithm guarantees that every vertex from V_i, $1 < i \le k$, has at least one neighbor in every V_j, $1 \le j < i$. If a vertex u from U_i has a neighboring vertex in every V_j, $1 \le j < i$, it is included in V_i. Since U_i is an independent set, none of the vertices included in V_i is adjacent to u. Thus, all sets of $\mathcal{P}$ are independent, while $|\mathcal{P}| = k$, completing the proof. □

The next theorem improves an upper bound for every graph G with $\Delta(G) \ge 5$, compared to Corollary 1.4.

Theorem 2.3. *For any graph G with $\Delta(G) \ge 5$,*

$$\chi''_a(G) \le 2\Delta(G) - 1$$

The proof of this theorem is deferred to Section 3. Lu et al. [3] proved that $\chi''_a(G) \le 7$ for any graph with maximum degree 4. This result, together with Theorem 2.3, implies the following:

Corollary 2.4. *For any graph G with $\Delta(G) \ge 4$, we have $\chi''_a(G) \le 2\Delta(G) - 1$.*

Please cite this article in press as: B. Vučković, An improved upper bound on the adjacent vertex distinguishing total chromatic number of graphs, Discrete Mathematics (2017), https://doi.org/10.1016/j.disc.2017.10.011.