Discrete Mathematics 341 (2018) 957–964

Contents lists available at ScienceDirect

# Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# On tiling the integers with 4-sets of the same gap sequence

Ilkyoo Choi [a,*], Junehyuk Jung [b], Minki Kim [c]

[a] Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Gyeonggi-do, Republic of Korea
[b] Department of Mathematics, Texas A&M University, TX, United States
[c] Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea

ARTICLE INFO

*Article history:*
Received 22 September 2016
Received in revised form 21 November 2017
Accepted 27 December 2017

*Keywords:*
Integer partitions
Gap sequence
Tilings

ABSTRACT

Partitioning a set into similar, if not, identical, parts is a fundamental research topic in combinatorics. The question of partitioning the integers in various ways has been considered throughout history. Given a set $\{x_1, \ldots, x_n\}$ of integers where $x_1 < \cdots < x_n$, let the *gap sequence* of this set be the unordered multiset $\{d_1, \ldots, d_{n-1}\} = \{x_{i+1} - x_i : i \in \{1, \ldots, n-1\}\}$. This paper addresses the following question, which was explicitly asked by Nakamigawa: can the set of integers be partitioned into sets with the same gap sequence? The question is known to be true for any set where the gap sequence has length at most two. This paper provides evidence that the question is true when the gap sequence has length three. Namely, we prove that given positive integers $p$ and $q$, there is a positive integer $r_0$ such that for all $r \geq r_0$, the set of integers can be partitioned into 4-sets with gap sequence $p, q, r$.



## 1. Introduction

Let $[n]$ denote the set $\{1, \ldots, n\}$ and let $[a, b]$ denote the set $\{a, \ldots, b\}$. Note that $[1, 0] = \emptyset$. In addition, let $d[a, b] + c$ denote the set $\{da + c, d(a+1) + c, \ldots, db + c\}$. An $n$-set is a set of size $n$.

Partitioning a set into similar, if not, identical, parts is a fundamental research topic in combinatorics. In the literature, it is typically said that $T$ *tiles* $S$ if the set $S$ can be partitioned into parts that are all "similar" to $T$ in some sense. For example, Golomb initiated the study of tilings of the checker board with "polyominoes" in 1954 [3], and it has attracted a vast audience of both mathematicians and non-mathematicians. See the book by Golomb [4] for recent developments regarding this particular problem.

The question of partitioning the integers $\mathbb{Z}$ (and the positive integers $\mathbb{Z}^+$) in various ways has been considered throughout history. For two sets $T$ and $S$ where $T \subseteq S$ and a group $G$ acting on $S$, we say that "$T$ tiles $S$ under $G$" if $S$ can be partitioned into copies that are obtainable from $T$ via $G$; namely, there is a subset $X$ of $G$ such that $S = \dot{\bigcup}_{\gamma \in X} \gamma(T)$. Tilings of $\mathbb{Z}$ and $\mathbb{Z}^+$ under translation have already been extensively studied [2,8]. It is known that a set $S$ of integers tiles $\mathbb{Z}^+$ under translation if and only if $S$ tiles some interval of $\mathbb{Z}$ under translation. In particular, a 3-set $S$ tiles $\mathbb{Z}^+$ under translation if and only if the elements of $S$ form an arithmetic progression.

It is easy to see that an arbitrary 2-set of integers tiles an interval of $\mathbb{Z}$ (and therefore tiles $\mathbb{Z}$) under translation, and there are 3-sets of integers that do not tile $\mathbb{Z}$ under translation. However, if both translation and reflection are allowed, then Sands and Świerczkowski [13] provided a short proof that an arbitrary 3-set of real numbers tiles $\mathbb{R}$ (simplifying a proof in [7]), and on the way they also proved that an arbitrary 3-set of integers tiles $\mathbb{Z}$. It is also known that not all 4-sets of integers tile $\mathbb{Z}$ under translation and reflection.

* Corresponding author.
*E-mail addresses:* ilkyoo@hufs.ac.kr (I. Choi), junehyuk@math.tamu.edu (J. Jung), kmk90@kaist.ac.kr (M. Kim).

https://doi.org/10.1016/j.disc.2017.12.017
0012-365X/

In his book [6], Honsberger strengthened the previous result with a simple greedy algorithm by showing that an arbitrary 3-set of integers tiles an interval of $\mathbb{Z}$ under translation and reflection. Meyerowitz [9] analyzed this algorithm and gave a constructive proof that the algorithm produces a tiling of an interval of $\mathbb{Z}$, and also proved that a 3-set of real numbers tiles $\mathbb{R}^+$, strengthening an aforementioned result. This algorithm does not necessarily find the shortest interval of $\mathbb{Z}$ that a 3-set of integers can tile; there has been effort in trying to determine the shortest such interval [1,10], and in some cases the shortest such interval is known.

Gordan [5] generalized the problem to higher dimensions. He proved that a 3-set of $\mathbb{Z}^n$ tiles $\mathbb{Z}^n$ under the Euclidean group actions (translation, reflection, and rotation), and that there is a set of size $4n - 2\lfloor n/2 \rfloor$ of $\mathbb{Z}^n$ that does not tile $\mathbb{Z}^n$ under the Euclidean group actions. More information regarding higher dimensions is in Section 4. There is also a paper [12] that studies tilings of the cyclic group $\mathbb{Z}_n$.

This paper focuses on partitioning $\mathbb{Z}$ into sets with the same "gap sequence" and "gap length", which is the term used in [12] and [11], respectively. Given a set $\{x_1, \dots, x_n\}$ of integers where $x_1 < \cdots < x_n$, let the *gap sequence* of this set be the unordered multiset $\{d_1, \dots, d_{n-1}\} = \{x_{i+1} - x_i : i \in [n-1]\}$. For convenience, we will say a set with gap sequence $\{d_1, \dots, d_{n-1}\}$ is a $(d_1, \dots, d_{n-1})$-*set*. Note that the gap sequence of a set with $n$ elements has length $n-1$. Roughly speaking, in addition to reflecting the order of the gaps of a given set, any permutation of the order of the gaps of the set is allowed.

In [11], the following question was explicitly asked:

**Question 1.1** (*[11]*). *For every gap sequence $S$ of length $n-1$, can $\mathbb{Z}$ be partitioned into $n$-sets with the same gap sequence $S$?*

The answer is clearly yes for $n = 1$ and $n = 2$. Since allowing permutations of the order of the gaps of a given set does not provide additional help (when reflections of the gaps are already allowed), the results above imply that the answer is also yes for $n = 3$. In this paper, we provide evidence that the answer is yes for $n = 4$. Recall that if $u_1$ and $u_2$ are positive integers with $\gcd(u_1, u_2) = 1$, then every integer $s \geq (u_1 - 1)(u_2 - 1)$ can be written as a linear combination of $u_1$ and $u_2$, namely, $s = c_1u_1 + c_2u_2$ for some non-negative integers $c_1$ and $c_2$. Corollary 1.3 is an immediate consequence of Theorem 1.2.

**Theorem 1.2.** *There is an interval of the integers that can be partitioned into 4-sets with the same gap sequence $p, q, r$, if $q \geq p$ and $r \geq \max\{4q(4q-1), \frac{1}{\gcd(p,q)}(5p + 4q - \gcd(p, q))(4p + 3q - \gcd(p, q))\}$.*

**Corollary 1.3.** *There is an interval of the integers that can be partitioned into 4-sets with the same gap sequence $p, q, r$, if $r \geq 63(\max\{p, q\})^2$.*

Note that for the sake of presentation, we omit some improvements on the constants of the threshold on $r$. Our proof follows the ideas in [10,11], where partitions of $\mathbb{Z}^2$ is used to aid the partition of $\mathbb{Z}$. We develop and push the method further and generalize it to $\mathbb{Z}^3$. In Section 2, we show that we can partition certain subsets of $\mathbb{Z}^3$ into smaller subsets of $\mathbb{Z}^3$ that we call *blocks*. In Section 3, we demonstrate how to use the lemmas in Section 2 to tile an interval of $\mathbb{Z}$ with 4-sets with the desired gap sequence. We finish the paper with some open questions in Section 4.

## 2. Lemmas

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ be the usual unit vectors in $\mathbb{Z}^3$. An ordered 4-tuple $\{v_1, v_2, v_3, v_4\}$ of vectors in $\mathbb{Z}^3$ is called a $(e_1, e_2, e_3)$-*block* if the vectors $v_2 - v_1, v_3 - v_2, v_4 - v_3$ are $e_1, e_2, e_3$ in some order. We similarly define a $(e_1, e_2 - e_1, e_3)$-*block*. Given an integer $w > 1$ and a vector $v_0 = (x_0, y_0, z_0)$, we refer to $(wx_0, y_0, z_0)$ as "$v_0$ stretched in the $e_1$ direction by $w$". We similarly refer to stretching a set of vectors. For example, an $(e_1, e_2 - e_1, e_3)$-block stretched in the $e_1$ direction by $w$ becomes a $(we_1, e_2 - we_1, e_3)$-block and still consists of 4 vectors.

A set $S \subset \mathbb{Z}^2$ is called a *base*. A base $S$ can be *covered* (with *height* $h(S)$) by $(e_1, e_2, e_3)$-blocks if there exists an integer $h(S)$ such that $S \times h(S) = \{(x, y, z) : (x, y) \in S, z \in [h(S)]\}$ can be partitioned into $(e_1, e_2, e_3)$-blocks. If $S$ can be covered with height $h(S)$ by $(e_1, e_2 - e_1, e_3)$-blocks, some stretched in the $e_1$ direction by $p$ and others by $q$, we say "it can be covered by $(pe_1, e_2 - pe_1, e_3)$-blocks and $(qe_1, e_2 - qe_1, e_3)$-blocks". There are obvious generalizations of these terms, but these are the only ones we use in the paper.

### 2.1. When $q \geq 2p$

**Lemma 2.1.** *The following sets of $\mathbb{Z}^2$ can be covered by $(e_1, e_2, e_3)$-blocks:*

(*i*) $S_1 = \{(1, 1), (1, 2), (2, 2)\}$ *with* $h(S_1) = 4$
(*ii*) $S_2 = \{(1, 1), (2, 1), (2, 2)\}$ *with* $h(S_2) = 4$
(*iii*) $S_3 = [3] \times [2]$ *with* $h(S_3) = 4$
(*iv*) $S_4 = [k] \times [4]$ *for* $k \geq 2$ *with* $h(S_4) = 20$
(*v*) $S_5 = ([2] \times [4]) \cup \{(3, 1), (3, 2)\}$ *with* $h(S_5) = 4$
(*vi*) $S_6 = ([2] \times [4]) \cup \{(3, 4)\}$ *with* $h(S_6) = 4$
(*vii*) $S_7 = ([k] \times [4]) \cup \{(k + 1, 4)\}$ *for* $k \geq 2$ *with* $h(S_7) = 20$.