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Completing the spectrum of almost resolvable cycle systems with odd cycle length

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ABSTRACT

In this paper, we construct almost resolvable cycle systems of order $4k+1$ for odd $k \geq 11$. This completes the proof of the existence of almost resolvable cycle systems with odd cycle length. As a by-product, some new solutions to the Hamilton–Waterloo problem are also obtained.

1. Introduction

In this paper, we use $V(H)$ and $E(H)$ to denote the vertex-set and the edge-set of a graph H, respectively. We denote the cycle of length k by C_k and the complete graph on v vertices by K_v. A *factor* of a graph H is a spanning subgraph whose vertex-set coincides with $V(H)$. If its connected components are isomorphic to G, we call it a *G-factor*. A *G-factorization* of H is a set of edge-disjoint G-factors of H whose edge-sets partition $E(H)$. A C_k-factorization of H is a partition of $E(H)$ into C_k-factors. An r-regular factor is called an *r-factor*. Also, a 2-*factorization* of a graph H is a partition of $E(H)$ into 2-factors.

A *k-cycle system* of order v is a collection of k-cycles which partition $E(K_v)$. A k-cycle system of order v exists if and only if $3 \leq k \leq v$, $v \equiv 1 \pmod 2$ and $v(v-1) \equiv 0 \pmod{2k}$ [2,8,27,32]. A k-cycle system of order v is *resolvable* if it has a C_k-factorization. A resolvable k-cycle system of order v exists if and only if $3 \leq k \leq v$, v and k are odd, and $v \equiv 0 \pmod k$, see [3,4,16,20,21,25,26]. If $v \equiv 1 \pmod{2k}$, then a k-cycle system exists, but it is not resolvable. In this case, Vanstone et al. [28] started the research of the existence of an almost resolvable k-cycle system.

In a k-cycle system of order v, a collection of $(v-1)/k$ disjoint k-cycles is called an *almost parallel class*. In a k-cycle system of order $v \equiv 1 \pmod{2k}$, the maximum possible number of almost parallel classes is $(v-1)/2$, in which case a half-parallel class containing $(v-1)/2k$ disjoint k-cycles is left over. A k-cycle system of order v whose cycle set can be partitioned into $(v-1)/2$ almost parallel classes and a half-parallel class is called an *almost resolvable k-cycle system*, denoted by k-ARCS(v).

For recursive constructions of almost resolvable k-cycle systems, C. C. Lindner, et al. [19] have considered the general existence problem of almost resolvable k-cycle system from the commutative quasigroup for $k \equiv 0 \pmod 2$ and made a hypothesis: if there exists a k-ARCS$(2k+1)$ for $k \equiv 0 \pmod 2$ and $k \geq 8$, then there exists a k-ARCS$(2kt+1)$ except possibly for $t = 2$. H. Cao et al. [13,23,31] continued to consider the recursive constructions of an almost resolvable k-cycle system for $k \equiv 1 \pmod 2$. Many authors contributed to the following known results.

Theorem 1.1 *([1,6,13–15,19,28]). Let $k \geq 3$, $t \geq 1$ be integers and $n = 2kt+1$. There exists a k-ARCS(n) for $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 14\}$, except for $(k,n) \in \{(3,7),(3,13),(4,9)\}$ and except possibly for $(k,n) \in \{(8,33),(14,57)\}$.*

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Theorem 1.2 ([23,31]). *For any odd $k \geq 11$, there exists a k-ARCS$(2kt+1)$, where $t \geq 1$ and $t \neq 2$.*

In this paper, we construct almost resolvable cycle systems of order $4k+1$ for odd $k \geq 11$. Combining the known results in Theorems 1.1–1.2, we will prove the following main result.

Theorem 1.3. *For any odd $k \geq 3$, there exists a k-ARCS$(2kt+1)$ for all $t \geq 1$ except for $(k,t) \in \{(3,1),(3,2)\}$.*

2. Preliminary

In this section we present a basic lemma for the construction of a k-ARCS$(4k+1)$. The main idea is to find some initial cycles with special properties such that all the required almost parallel classes can be obtained from them. We need the following notions for that lemma.

Suppose Γ is an additive group and $I = \{\infty_1, \infty_2, \ldots, \infty_f\}$ is a set which is disjoint with Γ. We will consider an action of Γ on $\Gamma \cup I$ which coincides with the right regular action on the elements of Γ, and the action of Γ on I will coincide with the identity map. In other words, for any $\gamma \in \Gamma$, we have that $x+\gamma$ is the image under γ of any $x \in \Gamma$, and $x+\gamma = x$ holds for any $x \in I$. Given a graph H with vertices in $\Gamma \cup I$, the *translate* of H by an element γ of Γ is the graph $H+\gamma$ obtained from H by replacing each vertex $x \in V(H)$ with the vertex $x+\gamma$. The *development* of H under a subgroup Σ of Γ is the collection $dev_\Sigma(H) = \{H+x \mid x \in \Sigma\}$ of all translates of H by an element of Σ.

For our constructions, we set $\Gamma = \mathbb{Z}_k \times \mathbb{Z}_4$. Given a graph H with vertices in Γ and any pair $(r,s) \in \mathbb{Z}_4 \times \mathbb{Z}_4$, we set $\Delta_{(r,s)}H = \{x-y \mid \{(x,r),(y,s)\} \in E(H)\}$. Finally, given a list $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ of graphs, we denote by $\Delta_{(r,s)}\mathcal{H} = \cup_{i=1}^{t} \Delta_{(r,s)}H_i$ the multiset union of the $\Delta_{(r,s)}H_i$s.

Lemma 2.1. *Let $v = 4k+1$ and $\mathcal{C} = \{F_1, F_2\}$ where each F_i $(i=1,2)$ is a vertex-disjoint union of four cycles of length k satisfying the following conditions:*

(i) $V(F_i) = ((\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\}) \setminus \{(a_i, b_i)\}$ *for some* $(a_i,b_i) \in \mathbb{Z}_k \times \mathbb{Z}_4$, $i = 1,2$;
(ii) ∞ *has a neighbor in* $\mathbb{Z}_k \times \{j\}$ *for each* $j \in \mathbb{Z}_4$;
(iii) $\Delta_{(p,p)}\mathcal{C} = \mathbb{Z}_k \setminus \{0\}$ *for each* $p \in \{0,1\}$;
(iv) $\Delta_{(q,q)}\mathcal{C} = \mathbb{Z}_k \setminus \{0, \pm d_q\}$ *for each* $q \in \{2,3\}$, *where* d_q *satisfies* $(d_q, k) = 1$;
(v) $\Delta_{(r,s)}\mathcal{C} = \mathbb{Z}_k$ *for each pair* $(r,s) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ *satisfying* $r \neq s$.

Then, there exists a k-ARCS(v).

Proof. Let $V(K_v) = (\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\}$. Note that $0, d_q, 2d_q, \ldots, (k-1)d_q$ are k distinct elements since $(d_q,k) = 1$. Then we have the required half parallel class which is formed by the two cycles $((0,q),(d_q,q),(2d_q,q),\ldots,((k-1)d_q,q))$, $q = 2,3$. By (i), we know that F_i is an almost parallel class. All the required $2k$ almost parallel classes are $F_i + (l,0)$, $i = 1,2$, $l \in \mathbb{Z}_k$.

Now we show that the half parallel class and the $2k$ almost parallel classes form a k-ARCS(v). Let F' be a graph with the edge-set $\{\{(a,q),(a+d_q,q)\} \mid a \in \mathbb{Z}_k, q = 2,3\}$ and $\Sigma := \mathbb{Z}_k \times \{0\}$. Let $\mathcal{F} = dev_\Sigma(\mathcal{C}) \cup F'$. The total number of edges – counted with their respective multiplicities – covered by the almost parallel classes and the half parallel class of $\mathcal{F}$ is $2k(4k+1)$, that is exactly the size of $E(K_v)$. Therefore, we only need to prove that every pair of vertices lies in a suitable translate of $\mathcal{C}$ or in F'. By (ii), an edge $\{(z,j),\infty\}$ of K_v must appear in a cycle of $dev_\Sigma(\mathcal{C})$.

Now consider an edge $\{(z,j),(z',j')\}$ of K_v whose vertices both belong to $\mathbb{Z}_k \times \mathbb{Z}_4$. If $j = j' \in \{2,3\}$ and $z - z' \in \{\pm d_q\}$, then this edge belongs to F'. In all other cases there is, by (iii)-(v), an edge of some F_i of the form $\{(w,j),(w',j')\}$ such that $w - w' = z - z'$. It then follows that $F_i + (-w'+z', 0)$ is an almost parallel class of $dev_\Sigma(F_i)$ containing the edge $\{(z,j),(z',j')\}$ and the conclusion follows. □

3. k-ARCS$(4k+1)$ for $k \equiv 1 \pmod 4$

In this section, we will prove the existence of a k-ARCS$(4k+1)$ for $k \equiv 1 \pmod 4$.

Lemma 3.1. *For any $k \geq 13$ and $k \equiv 1 \pmod 4$, there exists a k-ARCS$(4k+1)$.*

Proof. Let $v = 4k+1$ and $k = 4n+1$, $n \geq 3$. We use Lemma 2.1 to construct a k-ARCS(v) with $V(K_v) = (\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\}$. The required parameters in (i) and (iv) of Lemma 2.1 are $(a_1,b_1) = (0,3)$, $(a_2,b_2) = (0,2)$, $d_2 = 2$, and $d_3 = \frac{k-1}{2}$. The required 8 cycles in $F_1 = \{C_1, C_2, C_3, C_4\}$ and $F_2 = \{C_5, C_6, C_7, C_8\}$ are listed as below.

The cycle C_1 is the concatenation of the sequences T_1, $(0,0)$, and T_2, where

$T_1 = ((n,0),(-n,1),\ldots,\underline{(n-i,0),(-(n-i),1)},\ldots,(1,0),(-1,1))$, $0 \leq i \leq n-1$;

$T_2 = ((1,1),(-1,0),\ldots,\underline{(1+i,1),(-(1+i),0)},\ldots,(n,1),(-n,0))$, $0 \leq i \leq n-1$.

Note: Actually T_1 can be viewed as the concatenation of the sequences $T_1^0, T_1^1, \ldots, T_1^{n-1}$, where the general formula is $T_1^i = ((n-i,0),(-(n-i),1))$, $0 \leq i \leq n-1$. Thus, for brevity, we just list the first sequence at the beginning of T_1