Discrete Mathematics 341 (2018) 990–996

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A proof of Terras' conjecture on the radius of convergence of the Ihara zeta function

Seiken Saito

Waseda University, Shinjuku, Tokyo 169-8050, Japan

ARTICLE INFO

Article history:
Received 6 September 2017
Accepted 3 January 2018
Available online 25 January 2018

Keywords:
Spectral radius of graphs
Ihara zeta functions
Average degree of graphs

ABSTRACT

We prove the conjecture by Audrey Terras (2011) on the inequalities among the spectral radius ρ_X of a finite graph X, the radius of convergence R of its Ihara zeta function $Z_X(u)$, and the average degree $\overline{d_X}$ of X. Relating to Terras' conjecture, we propose a new conjecture between R and certain Rayleigh quotients.

1. Introduction

Let $X = (V, E)$ be a finite undirected graph with vertex set V, where $|V| = n$, and edge set E, where $|E| = m$. We assign an arbitrary direction o to the edges of X and denote the set of all directed edges by E^o. For a directed edge $a \in E^o$, let $\bar{a}$ be the inverse directed edge of a. Denote the set of all directed edges and all inverse directed edges by

$$\vec{E} := \{a, \bar{a} \ : \ a \in E^o\}.$$

We call the directed graph $\vec{X} = (V, \vec{E})$ the *symmetric digraph* of X. For a directed edge $a \in \vec{E}$, let $o(a)$ and $t(a)$ be the origin and the terminus of a, respectively. For a natural number ℓ, a sequence of ℓ directed edges $C = a_1 \cdots a_\ell$ ($a_i \in \vec{E}$) is called a *path* of X if $t(a_i) = o(a_{i+1})$ for all $i = 1, \ldots, \ell - 1$. The natural number $\ell = \ell(C)$ is called the *length* of C. A path $C = a_1 \cdots a_\ell$ is said to have a *backtrack* if $a_{j+1} = \bar{a}_j$ for some $j = 1, \ldots, \ell - 1$. A path $C = a_1 \cdots a_\ell$ is said to have a *tail* if $a_\ell = \bar{a}_1$. If a path $C = a_1 \cdots a_\ell$ satisfies $t(a_\ell) = o(a_1)$, then C is said to be a *closed path*. For a closed path $C = a_1 \cdots a_\ell$, we denote the equivalence class of its cyclic arrangements by $[C]$:

$$[C] := \{a_1 \cdots a_\ell, \ \ a_2 \cdots a_\ell a_1, \ \ \ldots, \ \ a_\ell a_1 \cdots a_{\ell-1}\}.$$

A closed path $P = a_1 \cdots a_\ell$ is called a *prime closed path* if it has no backtrack, no tail, and $P \neq D^f$ holds for each cycle D in X and for each integer $f \in \mathbf{Z}_{\geq 2}$. The equivalence class $[P]$ of a prime closed path P in X is called a *prime* in X. For a finite connected graph $X = (V, E)$, the *Ihara zeta function* $Z_X(u)$ of X is defined by

$$Z_X(u) := \prod_{[P]} (1 - u^{\ell(P)})^{-1}$$

for u with a sufficiently small absolute value. Here, the product runs over all primes $[P]$ in X. $Z_X(u)$ is the reciprocal of a certain polynomial. Specifically, $Z_X(u)^{-1}$ has the following determinant formulas:

$$\begin{aligned} Z_X(u)^{-1} &= (1 - u^2)^{m-n} \det(I_n - uA + u^2(D - I_n)) && \text{(the Ihara–Bass formula)} \\ &= \det(I_{2m} - uW) && \text{(the Bass–Hashimoto formula),} \end{aligned}$$

E-mail address: seiken.saitou@gmail.com.

https://doi.org/10.1016/j.disc.2018.01.005

where A is the adjacency matrix of X, $D := \mathrm{diag}(\deg(x))_{x\in V}$ is the degree matrix of X, and $W := (W_{ab})_{a,b\in\vec{E}}$ is called the *edge matrix* of $\vec{X}$ defined by

$$W_{ab} := \begin{cases} 1 & \text{if } t(a) = o(b) \text{ and } b \neq \bar{a}, \\ 0 & \text{otherwise.} \end{cases}$$

Because the entries of W are nonnegative and W is irreducible, the Perron–Frobenius theorem implies that the spectral radius $\rho(W)$ is a simple positive eigenvalue of W. Thus, by the Bass–Hashimoto formula, the radius of convergence of $Z_X(u)$ is $R = R_X := 1/\rho(W)$. Note that if X is not a cycle graph and has no degree-one vertices, then $0 < R < 1$ (see e.g., p. 197 of [3]). By analogy between the Riemann zeta function and the Ihara zeta function, $Z_X(u)$ satisfies the *graph theory Riemann Hypothesis (RH)* if and only if $Z_X(u)$ has no poles in the region $R < |u| < \sqrt{R}$ (see p. 53 of [3]). Spec(A) denotes the spectrum of A. Let $\rho_X = \rho(A) := \max\{|\lambda| : \lambda \in \mathrm{Spec}(A)\}$ be the spectral radius of A and let

$$\rho'_X := \max\{|\lambda| : \lambda \in \mathrm{Spec}(A),\ |\lambda| \neq \rho_X\}.$$

A connected d-regular graph X is said to be a *Ramanujan graph* if and only if

$$\rho'_X \leq 2\sqrt{d-1} \qquad \text{(the Ramanujan inequality).} \tag{1}$$

Ramanujan graphs are highly connected, but they have sparse edges (see e.g., Section 9.4 in [3]). Hence Ramanujan graphs have many good properties suitable for communication networks. Ramanujan graphs are defined only for regular graphs. How may the concept of Ramanujan graphs be extended to irregular graphs? That is, it is a natural question to ask what an "irregular Ramanujan graph" is. An important fact that may provide a hint in answering this question is the equivalence between the graph theory RH and the Ramanujan inequality (1). That is, if X is a connected d-regular graph, then $D = dI_n$ and by the Ihara–Bass formula, the following equivalence holds:

$$Z_X(u) \text{ satisfies the graph theory RH} \Longleftrightarrow X \text{ is a Ramanujan graph.}$$

By the above equivalence, the graph theory RH characterizes Ramanujan graphs for regular graphs. Because the graph theory RH is defined for irregular graphs, it is important to investigate the relation between the graph theory RH and a certain inequality of ρ'_X like the Ramanujan inequality (1) such that if X is a d-regular graph then it coincides (1). Terras cited the following inequalities (see Chapter 8 in [3]) as the candidates for the Ramanujan inequality of irregular graphs:

$$\rho'_X \leq 2\sqrt{\overline{d_X}-1} \qquad \text{(the Hoory inequality),} \tag{2}$$

$$\rho'_X \leq 2\sqrt{\rho_X-1} \qquad \text{(the naive Ramanujan inequality),} \tag{3}$$

$$\rho'_X \leq \sigma_X \qquad \text{(the Lubotzky inequality),} \tag{4}$$

where $\overline{d_X} := \frac{1}{n}\sum_{x\in V}\deg(x)$ is the average degree of X and σ_X is the spectral radius of the adjacency operator on the universal covering tree of X. Since the inequality $2\sqrt{\overline{d_X}-1} \leq \sigma_X$ was proved by Hoory (see Corollary 2 [1]), (2) $\Longrightarrow$ (4) holds. Terras investigated several graphs to check whether the inequalities (2), (3), or the graph theory RH holds (see Examples 8.5–8.7 in [3]). Note that $R = 1/(d-1)$ when X is a d-regular graph. The following inequality may also be a candidate for the Ramanujan inequality of irregular graphs:

$$\rho'_X \leq \frac{2}{\sqrt{R}}. \tag{5}$$

In order to compare the inequalities (2), (3) and (5), we need the inequalities among the constants $\overline{d_X}$, ρ_X and $1+1/R$. Terras did not state (5) in [3]. However, it is natural to think that Terras considered (5) and compared these three constants. By computation of examples, Terras states the following conjecture in [3] (see Inequalities (8.9), p. 54):

Conjecture 1 (*A. Terras*)**.** *Let $X = (V, E)$ be a finite connected graph with vertex set V, with $|V| = n$, and edge set E, with $|E| = m$. Suppose that X is not a cycle graph (i.e., $X \not\simeq C_k$ for all $k \in \mathbf{Z}_{\geq 1}$) and X has no degree-one vertices. Let A be the adjacency matrix of X, and let ρ_X be the spectral radius of A. Let $\overline{d_X}$ be the average degree of X. Let R be the radius of convergence of the Ihara zeta function $Z_X(u)$ of X. Then, the following inequalities hold:*

$$\rho_X \geq 1 + \frac{1}{R} \geq \overline{d_X}. \tag{6}$$

Note that if X is a d-regular graph, then $\rho_X = \overline{d_X} = d$ and $R = 1/(d-1)$. Thus, the equalities in (6) hold.

We remark that Terras did not explicitly write the assumptions of the conjecture. However, she wrote "Suppose that a graph X satisfies our usual hypotheses (see Section 2.1)" in Theorem 8.1 in [3]. The graphs considered in the discussions of Chapter 8 in [3] satisfy the assumptions of Theorem 8.1. Thus, the explicit form of the conjecture proposed by Terras is Conjecture 1.

Relating to (6), Terras proved the weak inequality

$$\rho_X \geq \frac{p}{q} + \frac{1}{R},$$