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Graphs with few total dominating sets

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ABSTRACT

We give a lower bound for the number of total dominating sets of a graph together with a characterization of the extremal graphs, for trees as well as arbitrary connected graphs of given order. Moreover, we obtain a sharp lower bound involving both the order and the total domination number, and characterize the extremal graphs as well.

1. Introduction

A dominating set D is a set of vertices of a graph G with the property that every vertex of G either lies in D or has a neighbor in D. Total domination is an even stronger property: D is called a total dominating set of G if *every* vertex of G has a neighbor in D, whether or not it lies in D itself. The most classical and well-studied graph parameter in the context of (total) domination is the *(total) domination number*: the domination number $\gamma(G)$ of a graph G is the smallest cardinality of a dominating set, and likewise the total domination number $\gamma_t(G)$ is the smallest cardinality of a total dominating set. Numerous upper and lower bounds and other results on these numbers have been obtained over the years, we refer to the books [6,7] by Haynes, Hedetniemi and Slater and the more recent book [8] by Henning and Yeo, which focuses on total domination, for a comprehensive treatment of the subject. Comparatively little work has been done on the *number* of dominating or total dominating sets. These belong to the class of graph parameters based on counting subsets with specific properties; other well-studied examples are the number of independent sets and the number of matchings, see [12] for a recent survey including these and other examples.

As for the number $\partial(G)$ of dominating sets of a graph G, one has the trivial bounds

$$1 \le \partial(G) \le 2^n - 1,$$

with equality for the empty and complete graphs, respectively. Bród and Skupień [2] studied the number of dominating sets in trees. The maximum of $2^{n-1} + 1$ for trees with n vertices is attained only by the star (except for the cases $n = 4$ and $n = 5$, when the path also attains the maximum). The lower bound, on the other hand, is not only more complicated, the trees attaining it are no longer unique. We will observe a similar phenomenon for total domination in this paper. As shown in [13], the lower bound for trees is also sharp for arbitrary connected graphs and even graphs without isolated vertices. See also the recent paper by Skupień [11]. For trees, similar results can also be found for the number of efficient dominating sets,

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minimal dominating sets and minimal 2-dominating sets (see [3,9,10]). In these cases, the maximum is more interesting, though.

The focus of this paper is the number of total dominating sets of a graph, which we will denote by $\partial_t(G)$. We have similarly trivial bounds:

$$0 \le \partial_t(G) \le 2^n - n - 1.$$

We set $\partial_t(G) = 0$ if G has one or more isolated vertices, since in this case the graph does not have any total dominating sets. This is the only case in which the lower bound holds with equality. The upper bound holds with equality if and only if G is a complete graph. For trees, the upper bound is still quite simple, but it already illustrates some of our main ideas:

Proposition 1. *We have*

$$\partial_t(T) \le 2^{n-1} - 1$$

for every tree T with n vertices, with equality only for the star.

Proof. The statement is trivial for $n = 1$ and $n = 2$, so let us assume that $n \ge 3$. Every tree T with at least three vertices has two or more leaves, thus at least one vertex adjacent to a leaf (we will call such a vertex a support vertex); we denote this vertex by v. The vertex v has to be part of every total dominating set. This leaves us with only 2^{n-1} possible sets remaining, of which $\{v\}$ is clearly not a total dominating set (v is not dominated). Thus $\partial_t(T) \le 2^{n-1} - 1$, and equality can only hold if v is the only vertex adjacent to a leaf. This only holds for the star. ■

Just as for the number of dominating sets, the lower bound is more interesting. In the following section, we will show that the minimum number of total dominating sets of a tree (connected graph, or even arbitrary graph without components of order 1 or 2) with n vertices is of order $\Theta(9^{n/7})$. A precise bound, along with the characterization of the extremal graphs, is given in Theorems 5, 6 and 7. In Theorems 16 and 17 we obtain a sharp lower bound for $\partial_t(G)$ that takes the total domination number into account as well. See Section 3 for details.

2. The general lower bound

In this section, we determine the minimum number of total dominating sets of a connected graph with n vertices for arbitrary n. In fact, we will show that the lower bound we obtain remains valid for disconnected graphs as long as we exclude trivial components of one vertex (for which there is no total dominating set) or two vertices (for which the only total dominating set consists of both vertices).

It turns out to be advantageous to prove the lower bound for trees first (Theorem 5), and to generalize it to connected graphs (Theorem 6) and arbitrary graphs (Theorem 7) later. *Leaves* and *support vertices* will play an important role: we call a vertex with only a single neighbor a leaf, even if the graph is not a tree. The unique neighbor of a leaf is called a support vertex. The trivial observation that every support vertex has to be contained in every total dominating set of a graph will become very useful in the following.

As we will see, the extremal graphs are obtained as unions of *subdivided stars*: the subdivided star $S(K_{1,r})$ is obtained from a star $K_{1,r}$ with r leaves by subdividing each edge into two edges (thus introducing an additional vertex on each edge).

Let us now begin our discussion by considering trees. We will write $m_n = \min\{\partial_t(T) : |T| = n\}$ for the minimum number of total dominating sets of a tree with n vertices, and $\mathcal{T}_n = \{T : |T| = n, \ \partial_t(T) = m_n\}$ for the set of all trees that attain this minimum. We start with a very useful lemma on merging trees.

Lemma 2. *Let T_1 and T_2 be two trees, and let v_1, v_2 be vertices of T_1 and T_2, respectively. Consider the tree T obtained by adding the edge v_1v_2 to the union $T_1 \cup T_2$. We have*

$$\partial_t(T) \ge \partial_t(T_1)\partial_t(T_2),$$

and the equality holds if and only if v_1 and v_2 are at distance 2 from a leaf in T_1 and T_2, respectively (v_1 and v_2 may themselves be leaves).

Proof. Obviously, every total dominating set of $T_1 \cup T_2$ is also a total dominating set of T, which readily proves the inequality: note that $\partial_t(T_1 \cup T_2) = \partial_t(T_1)\partial_t(T_2)$, since every total dominating set of $T_1 \cup T_2$ is the union of a total dominating set of T_1 and a total dominating set of T_2, and vice versa. It remains to determine the cases of equality.

Assume first that both v_1 and v_2 have the required property. Consider any total dominating set D of T, and assume that its restriction to T_1 is not total dominating. The only reason this could happen is that v_1 is dominated by v_2 in T, but not by any other neighbor. One of these neighbors, however, is a support vertex in T_1 by our assumption. If this support vertex is not present in D, then its leaf neighbor is not dominated, and we reach a contradiction. Thus the only total dominating sets of T are obtained as unions of total dominating sets of T_1 and T_2.

For the converse suppose, for instance, that there is no leaf in T_1 whose distance to v_1 is 2. Then none of v_1's neighbors in T_1 is a support vertex in T. It is easy to verify in this case that $D = V(T) \setminus N_{T_1}(v_1)$ (i.e., the set of all vertices of T except for v_1's neighbors in T_1) is a total dominating set of T, but the restriction of D to T_1 is clearly not (v_1 is not dominated). Thus $\partial_t(T) > \partial_t(T_1)\partial_t(T_2)$ in this case, and by symmetry the same argument applies to v_2. ■