Discrete Mathematics 341 (2018) 1021–1031

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Characterization of the induced matching extendable graphs with $2n$ vertices and $3n$ edges

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ARTICLE INFO

Article history:
Received 15 August 2017
Received in revised form 16 November 2017
Accepted 21 December 2017
Available online 25 January 2018

Keywords:
Matching
Perfect matching
Induced matching
Induced matching extendable

ABSTRACT

A graph G is induced matching extendable or IM-extendable if every induced matching of G is contained in a perfect matching of G. In 1998, Yuan proved that a connected IM-extendable graph on $2n$ vertices has at least $3n-2$ edges, and that the only IM-extendable graph with $2n$ vertices and $3n-2$ edges is $T \times K_2$, where T is an arbitrary tree on n vertices. In 2005, Zhou and Yuan proved that the only IM-extendable graph with $2n \geq 6$ vertices and $3n-1$ edges is $T \times K_2 + e$, where T is an arbitrary tree on n vertices and e is an edge connecting two vertices that lie in different copies of T and have distance 3 between them in $T \times K_2$. In this paper, we introduced the definition of Q-joint graph and characterized the connected IM-extendable graphs with $2n \geq 4$ vertices and $3n$ edges.

1. Introduction

Graphs considered in this paper are finite and simple. For notation and terminology not defined here, we follow [1]. For a graph G, its *vertex set* and *edge set* are denoted by $V(G)$ and $E(G)$, respectively. For any two vertices $u, v \in V(G)$, let $d_G(u, v)$ denote the length of a shortest path between u and v in G. For any vertex $v \in V(G)$, edge set $M \subseteq E(G)$ and vertex set $S \subseteq V(G)$, define $N_G(v) = \{u \in V(G) \setminus \{v\} : uv \in E(G)\}$, $N_G^2(v) = \{u : d_G(u, v) = 2\}$, $N_G(S) = \{u \in V(G) \setminus S : uw \in E(G) \text{ for some } w \in S\}$, $V(M) = \{u \in V(G) : \text{ there exists } w \in V(G) \text{ such that } uw \in M\}$ and $E(S) = \{uw \in E(G) : u, w \in S\}$, respectively. A set of edges $M \subseteq E(G)$ is a *matching* of G if for any two distinct elements $e, f \in M$, $V(e) \cap V(f) = \emptyset$. A matching M is *perfect* [1] if $V(M) = V(G)$. A matching M is *induced* [2] if $E(V(M)) = M$. A graph G is *induced matching extendable* or *IM-extendable* [12] if every induced matching of G is contained in a perfect matching of G.

A lot of researches on IM-extendable have been done since the definition of IM-extendable graph was proposed in [12]. The following are some of the topics which have been investigated: degree conditions of IM-extendable graphs [3], induced matching extendable graph powers [4], degree sum conditions of IM-extendable graphs [5], maximal IM-extendable graphs [6], 4-regular claw-free IM-extendable graphs [7], edge-deletable IM-extendable graphs with minimum number of edges [8], induced matching extendable graphs without K_4 minor [9], IM-extendable claw-free graphs [10], NP-completeness of induced matching problem and co-NP-completeness of induced matching extendable problem [11], degree conditions of induced matching extendable bipartite graphs [13], induced matching extendability of claw-free graphs of diameter 2 [14], and induced matching extendable graphs with $2n$ vertices and $3n-1$ edges in [15].

The *product* of T and K_2, denoted by $T \times K_2$, is the graph with vertex set $V(T \times K_2) = \{x_i : x \in V(T), i = 1, 2\}$ and edge set $E(T \times K_2) = \{x_1x_2 : x \in V(T)\} \cup \{x_iy_i : xy \in E(T), i = 1, 2\}$. For $i = 1, 2$, the subgraph $T(i)$ of $T \times K_2$ induced by $\{x_i : x \in V(T)\}$ is called the ith copy of T. Throughout this paper, x_i always denotes the vertex in $T(i)$ corresponding to the vertex x in T. In 1998, Yuan [12] proved that a connected IM-extendable graph on $2n$ vertices has at least $3n-2$ edges, and that the only IM-extendable graph with $2n$ vertices and $3n-2$ edges is $T \times K_2$, where T is an arbitrary tree on n vertices. In 2005, Zhou

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https://doi.org/10.1016/j.disc.2017.12.013

and Yuan [15] proved that the only IM-extendable graph with $2n \geq 6$ vertices and $3n-1$ edges is $T \times K_2 + e$, where T is an arbitrary tree on n vertices and e is an edge connecting two vertices that lie in different copies of T and have distance 3 between them in $T \times K_2$.

In Section 2, we listed some related known results which will be useful for the proof of our main results. In Section 3, we introduced the definition of double braced fat tree and Q-joint graph and characterized the connected IM-extendable graphs with $2n \geq 4$ vertices and $3n$ edges.

2. Some related known results

The following are some important results about IM-extendable graphs proved in [12] or [15].

Theorem 2.1 ([12]). *If G is a connected IM-extendable graph with $|V(G)| \geq 4$, then $|V(G)|$ is even and G is 2-connected.*

Theorem 2.2 ([12]). *For every graph G and every positive integer r, $G \times K_{2r}$ is IM-extendable.*

Theorem 2.3 ([12]). *A graph G is IM-extendable if and only if for every induced matching M of G and every $S \subseteq V(G)\backslash V(M)$, $o(G-V(M)-S) \leq |S|$.*

Let $D_2(G)$ denote the set of vertices of degree 2 in G, that is, $D_2(G) = \{u \in V(G) : d_G(u) = 2\}$. If $u \in D_2(G)$ with $N_G(u) = \{v, w\}$, then $G^* = G \bullet \{u, v, w\}$ is defined to be the simple graph obtained from G by identifying u, v, w to a new vertex u^*.

Theorem 2.4 ([12]). *Let $u \in D_2(G)$ with $N_G(u) = \{v, w\}$. If G is IM-extendable, then the simple graph $G^* = G \bullet \{u, v, w\}$ is IM-extendable.*

Theorem 2.5 ([12]). *Let $u, v \in D_2(G)$ such that $uv \in E(G)$. If G is IM-extendable, then $G - \{u, v\}$ is IM-extendable.*

A fat tree [15] is a graph isomorphic to $T \times K_2$, where T denotes a tree. *A braced fat tree* [15] is a graph isomorphic to $T \times K_2 + e$, where T denotes a tree and e is an edge connecting two vertices that lie in different copies of T and has distance 3 between them in $T \times K_2$.

Theorem 2.6 ([12]). *If G is a connected IM-extendable graph with $2n$ vertices, then $|E(G)| \geq 3n-2$, where equality holds if and only if G is a fat tree.*

Theorem 2.7 ([15]). *If G is a connected graph with $2n \geq 6$ vertices and $3n-1$ edges, then G is IM-extendable if and only if G is a braced fat tree.*

Theorem 2.8 ([12]). *The only 3-regular connected IM-extendable graphs are $C_n \times K_2$ for $n \geq 3$ and $C_{2n}(1, n)$ for $n \geq 2$.*

For any connected IM-extendable graph G, the following results are implied in [12] on page 207–209.

Lemma 2.9. *Let $u \in D_2(G)$ with $N_G(u) = \{v, w\}$. Then $vw \notin E(G)$.*

Lemma 2.10. *Let $u \in D_2(G)$ with $N_G(u) = \{v, w\}$. Then we have $v'w' \in E(G)$ for any $v' \in N_G(v)\backslash N_G(w)$ and $w' \in N_G(w)\backslash N_G(v)$.*

Lemma 2.11. *Let $u, v \in D_2(G)$ with $uv \in E(G)$. If $G - \{u, v\}$ is a fat tree $T \times K_2$, then $N_G(\{u, v\}) = \{x_1, x_2\}$ for a vertex $x \in V(T)$.*

Lemma 2.12. *Let $S_1 = \{u \in D_2(G) :$ there exists $u' \in N_G(u)$ such that $d_G(u') \geq 4\}$ and $S_2 = \{u \in D_2(G) :$ for all $u' \in N_G(u), d_G(u') = 3\}$. If $E(D_2(G)) = \emptyset$ and for any $u \in D_2(G)$ with $N_G(u) = \{v, w\}$, $N_G(v) \cap N_G(w) = \{u\}$, then $|E(G)| \geq \frac{3}{2}(|V(G)| + |S_2|)$.*

The following result is implied in [15] on pages 261–262.

Lemma 2.13. *Let $u, v \in D_2(G)$ with $uv \in E(G)$. If $G - \{u, v\}$ is a braced fat tree $T \times K_2 + e$, then one of the following is true and so G is a braced fat tree.*

(i) *$N_G(\{u, v\}) = \{x_1, x_2\}$ for a vertex $x \in V(T)$.*

(ii) *$N_G(\{u, v\}) = \{a_1, b_1\}$ or $N_G(\{u, v\}) = \{a_2, b_2\}$ only when $P = abc$, $d_T(a) = 1$, $d_T(b) = 2$ and $e = a_1c_2$.*

(iii) *$N_G(\{u, v\}) = \{b_1, c_1\}$ or $N_G(\{u, v\}) = \{b_2, c_2\}$ only when $P = abc$, $d_T(c) = 1$, $d_T(b) = 2$ and $e = a_1c_2$.*