Discrete Mathematics 341 (2018) 579–587

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

An update on non-Hamiltonian $\frac{5}{4}$-tough maximal planar graphs

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ARTICLE INFO

Article history:
Received 26 May 2017
Received in revised form 30 October 2017
Accepted 31 October 2017

Keywords:
Toughness
Shortness exponent
Maximal planar graphs

ABSTRACT

Studying the shortness of longest cycles in maximal planar graphs, we improve the upper bound on the shortness exponent of the class of $\frac{5}{4}$-tough maximal planar graphs presented by Harant and Owens (1995). In addition, we present two generalizations of a similar result of Tkáč who considered 1-tough maximal planar graphs (Tkáč, 1996); we remark that one of these generalizations gives a tight upper bound. We fix a problematic argument used in both mentioned papers.

1. Introduction

We continue the study of non-Hamiltonian graphs with the property that removing an arbitrary set of vertices disconnects the graph into a relatively small number of components (compared to the size of the removed set). In the present paper, we construct families of maximal planar such graphs whose longest cycles are short (compared to the order of the graph).

More formally, the properties which we study are the toughness of graphs and the shortness exponent of classes of graphs (both introduced in 1973). We recall that following Chvátal [5], the *toughness* of a graph G is the minimum, taken over all separating sets X of vertices in G, of the ratio of $|X|$ to $c(G-X)$ where $c(G-X)$ denotes the number of components of the graph $G-X$. The toughness of a complete graph is defined as being infinite. We say that a graph is t-*tough* if its toughness is at least t.

Along with the definition of toughness, Chvátal [5] conjectured that there is a constant t_0 such that every t_0-tough graph (on at least three vertices) is Hamiltonian. As a lower bound on t_0, Bauer et al. [2] presented graphs with toughness arbitrarily close to $\frac{9}{4}$ which contain no Hamilton path (and thus, they are non-Hamiltonian). While remaining open for general graphs, Chvátal's conjecture was confirmed in several restricted classes of graphs; and also various relations among the toughness of a graph and properties of its cycles are known. We refer the reader to the extensive survey on this topic [1].

Clearly, every graph (on at least five vertices) of toughness greater than $\frac{3}{2}$ is 4-connected, so every such planar graph is Hamiltonian by the classical result of Tutte [14]. On the other hand, Harant [8] showed that not every $\frac{3}{2}$-tough planar graph is Hamiltonian; and furthermore, the shortness exponent of the class of $\frac{3}{2}$-tough planar graphs is less than 1.

We recall that following Grünbaum and Walther [7], the *shortness exponent* of a class of graphs Γ is the lim inf, taken over all infinite sequences G_n of non-isomorphic graphs of Γ (for n going to infinity), of the logarithm of the length of a longest cycle in G_n to base equal to the order of G_n.

Introducing this notation, Grünbaum and Walther [7] also presented upper bounds on the shortness exponent for numerous subclasses of the class of 3-connected planar graphs. Furthermore, they remarked that the upper bound for

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https://doi.org/10.1016/j.disc.2017.10.034
0012-365X/

the class of 3-connected planar graphs itself was presented earlier by Moser and Moon [10] who used a slightly different notation. Later, Chen and Yu [4] showed that every 3-connected planar graph G contains a cycle of length at least $|V(G)|^{\log_3 2}$; in combination with the bound of [10], it follows that the shortness exponent of this class equals $\log_3 2$. A number of results considering the shortness exponent and similar parameters are surveyed in [12].

Considering the class of maximal planar graphs under a certain toughness restriction, Owens [11] presented non-Hamiltonian maximal planar graphs of toughness arbitrarily close to $\frac{3}{2}$. Harant and Owens [9] argued that the shortness exponent of the class of $\frac{5}{4}$-tough maximal planar graphs is at most $\log_9 8$. Improving the bound $\log_7 6$ presented by Dillencourt [6], Tkáč [13] showed that it is at most $\log_6 5$ for the class of 1-tough maximal planar graphs.

In the present paper, we show the following.

Theorem 1. *Let σ be the shortness exponent of the class of maximal planar graphs under a certain toughness restriction.*

(i) *If the graphs are $\frac{5}{4}$-tough, then σ is at most $\log_{30} 22$.*
(ii) *If the graphs are $\frac{8}{7}$-tough, then σ is at most $\log_6 5$.*
(iii) *If the toughness of the graphs is greater than 1, then σ equals $\log_3 2$.*

We note that $\log_9 8 > \log_{30} 22$, that is, the statement in item (i) of Theorem 1 improves the result of [9]. Furthermore, items (*ii*) and (*iii*) provide two different generalizations of the result of [13] since $\frac{8}{7} > 1$ and $\log_6 5 > \log_3 2$.

We remark that we fix a problem in a technical lemma presented in [9, Lemma 1]. The fixed version of this lemma (see Lemma 12) is applied to prove the present results.

2. Structure of the proof

In order to prove Theorem 1, we shall construct three families of graphs whose properties are summarized in the following proposition.

Proposition 2. *For every $i = 1, 2, 3$ and every non-negative integer n, there exists a maximal planar graph $F_{i,n}$ on $f_i(n)$ vertices whose longest cycle has $c_i(n)$ vertices where*

(i) $f_1(n) = 1 + 101(1 + 30 + \cdots + 30^n)$ *and* $c_1(n) = 1 + 93(1 + 22 + \cdots + 22^n)$ *and* $F_{1,n}$ *is* $\frac{5}{4}$*-tough,*
(ii) $f_2(n) = 1 + 14(1 + 6 + \cdots + 6^n)$ *and* $c_2(n) = 1 + 13(1 + 5 + \cdots + 5^n)$ *and* $F_{2,n}$ *is* $\frac{8}{7}$*-tough,*
(iii) $f_3(n) = 4 + 5(1 + 3 + \cdots + 3^n)$ *and* $c_3(n) = 3 \cdot 2^{n+3} - 9n - 15$ *and the toughness of* $F_{3,n}$ *is greater than* 1.

Before constructing the graphs $F_{1,n}$, we point out that the use of Proposition 2 leads directly to the main results of the present paper.

Proof of Theorem 1. We consider an infinite sequence of non-isomorphic graphs $F_{1,n}$ given by item (i) of Proposition 2; and we recall that they are $\frac{5}{4}$-tough maximal planar graphs. Furthermore, we have

$$f_1(n) = 1 + \tfrac{101}{29}(30^{n+1} - 1) \quad \text{and} \quad c_1(n) = 1 + \tfrac{93}{21}(22^{n+1} - 1).$$

It follows that

$$\lim_{n\to\infty} \log_{f_1(n)} c_1(n) = \log_{30} 22.$$

Thus, the considered shortness exponent is at most $\log_{30} 22$.

Using similar arguments and considering items (*ii*) and (*iii*) of Proposition 2, we obtain the desired upper bounds.

Clearly, if G is a maximal planar graph (on at least four vertices), then it is 3-connected. By a result of [4], G contains a cycle of length at least $|V(G)|^{\log_3 2}$. In combination with the upper bound obtained due to item (*iii*) of Proposition 2, we obtain that for the class of maximal planar graphs of toughness greater than 1, the shortness exponent equals $\log_3 2$. □

In the remainder of the present paper, we construct the families of graphs having the properties described in Proposition 2. Basically, we proceed in four steps. First, we introduce relatively small graphs $F_{i,0}$ called 'building blocks' in Section 3, and we observe key properties of their longest cycles. We use these building blocks to construct larger graphs $F_{i,n}$ in Section 4, and we show that their longest cycles are short. In Section 5, we study the toughness of the building blocks. The toughness of the graphs $F_{i,n}$ is shown in Sections 6 and 7.

We remark that the graphs $F_{1,n}$ and $F_{2,n}$ are obtained using a standard construction for bounding the shortness exponent (see for instance [7,6,9,13] or [3]); the improvement of the known bounds comes with the choice of suitable building blocks. In addition, we formalize the key ideas of this construction to make them more accessible for further usage.

The construction of graphs $F_{3,n}$ can be viewed as a simple modification of the construction used in [10] (yet the toughness and longest cycles of the constructed graphs are different).