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Planar graphs without 3-cycles adjacent to cycles of length 3 or 5 are (3, 1)-colorable

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ABSTRACT

Given a nonnegative integer d and a positive integer k, a graph G is said to be (k, d)-colorable if the vertices of G can be colored with k colors such that every vertex has at most d neighbors receiving the same color as itself. Let $\mathscr{F}$ be the family of planar graphs without 3-cycles adjacent to cycles of length 3 or 5. This paper proves that everyone in $\mathscr{F}$ is $(3, 1)$-colorable. This is the best possible in the sense that there are members in $\mathscr{F}$ which are not $(3, 0)$-colorable.

1. Introduction

Let k be a positive integer and d a nonnegative integer. A (k, d)-*coloring* of a graph G is a mapping $\psi : V(G) \longmapsto \{1, 2, \ldots, k\}$ such that every vertex $v \in V(G)$ has at most d neighbors receiving the same color as itself. A graph G is called (k, d)-*colorable* if it admits a (k, d)-coloring. The (k, d)-colorings are called defective or improper colorings in the earlier papers. Clearly, the $(k, 0)$-coloring is just the classical proper k-coloring. The $(k, 1)$-coloring, as observed in [17,16], offers an approach decomposing a graph into a matching and a k-colorable graph.

In terms of (k, d)-coloring, the well-known Four Color Theorem [1] states that every planar graph is $(4, 0)$-colorable. What happens if we color planar graphs with only three colors? Cowen, Cowen and Woodall [7] proved that all planar graphs are $(3, 2)$-colorable and there exist planar graphs which are not $(3, 1)$-colorable. Which condition can guarantee a planar graph to be $(3, 1)$- or even $(3, 0)$-colorable? Let C_k denote a cycle of length k. It is interesting to notice that every C_3-free planar graph is $(3, 0)$-colorable [9] while for every integer $k \geq 4$, there are C_k-free planar graphs which are not $(3, 0)$-colorable [5]. Since the Four Color Conjecture turned into Four Color Theorem in 1977 [1], the $(3, 0)$-colorability of planar graphs has been extensively studied in the literature. We refer the readers to [3] for a good overview on the study of $(3, 0)$-colorability of planar graphs. The most important two issues on the $(3, 0)$-colorability of planar graphs are to prove or disprove the following

Steinberg's conjecture (**1976** [2,11,13]): *Every planar graph with neither 4- nor 5-cycles is* (3,0)*-colorable*;
and to solve the following

Havel's problem (**1970** [10,11]): *Does there exist a constant C such that every planar graph with triangles at distance at least C is* (3,0)*-colorable?*

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In 2003, motivated by Steinberg's conjecture and Havel's problem, Borodin and Raspaud [5] proved that every planar graph with neither 5-cycles nor triangles at distance less than 4 is $(3, 0)$-colorable and proposed the following conjecture.

Bordeaux conjecture.
(1) *Every planar graph with neither* 5*-cycles nor intersecting triangles is* $(3, 0)$*-colorable* (the weak version).
(2) *Every planar graph with neither* 5*-cycles nor adjacent triangles is* $(3, 0)$*-colorable* (the strong version).

In 2009, after proving that planar graphs without triangles adjacent to cycles of length from 3 to 9 are $(3, 0)$-colorable, Borodin et al. [4] proposed the following conjecture.

Nsk3CC. *Every planar graph without* 3*-cycles adjacent to cycles of length* 3 *or* 5 *is* (3,0)*-colorable.*

Havel's problem has now been solved by Dvořák, Král and Thomas [14] in the positive. As for Steinberg's conjecture, very recently, Cohen-Addad et al. [6] disproved it by constructing a non-3-colorable planar graph with neither 4- nor 5-cycles. Since Nsk3CC is stronger than the stronger Bordeaux conjecture, and the later is again stronger than Steinberg's conjecture, Cohen-Addad et al. [6] actually disproved all conjectures above except the weak Bordeaux conjecture!

Nevertheless, one may raise the following problems.

Problem 1. Is every planar graph with neither 4- nor 5-cycles (3,1)-colorable?

Problem 2. Is every planar graph with neither 5-cycles nor adjacent triangles (3,1)-colorable?

Problem 3. Is every planar graph without 3-cycles adjacent to cycles of length 3 or 5 (3,1)-colorable?

Problem 1 has already been solved affirmatively in the list version by Lih et al. [12]. More precisely, in [12], the authors actually proved that, for each $i \in \{5, 6, 7\}$, planar graphs with neither 4- nor i-cycles are list $(3, 1)$-colorable. Later, Dong and Xu [8] extended the result to $i \in \{8, 9\}$; and in 2013, Wang and Xu [15] proved that planar graphs without 4-cycles are list $(3, 1)$-colorable.

Problem 2 has also been solved affirmatively by Xu [17]. In 2014, Wang and Xu [16] even proved that planar graphs without 5-cycles are $(3, 1)$-colorable.

Yet Problem 3 is still open up to date. This paper will solve Problem 3 affirmatively, too. Namely this paper will prove the following result.

Theorem 1. *Planar graphs without 3-cycles adjacent to cycles of length 3 or 5 are (3, 0)-colorable.*

The rest of this section is devoted to some definitions. Graphs considered here are finite, simple (i.e, no loops or multi-edges) and undirected. We follow [2] for those used but undefined notation and terminology. A graph G is *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such particular embedding of a planar graph is called a *plane* graph. Let $G = (V, E, F)$ be a graph with vertex set V, edge set E and face set F. For a vertex $v \in V$, the *degree* and the *neighborhood* of v are denoted by $d(v)$ and $N(v)$, respectively. Call a vertex $v \in V$ a *k-vertex* (resp. a *k^+-vertex* or a *k^--vertex*) if $d(v) = k$ (resp. $d(v) \geq k$ or $d(v) \leq k$). For a face $f \in F$, the set of vertices on f and the *boundary* walk of f are denoted by $V(f)$ and $b(f)$, respectively. The *size*, or more preferably here, the *degree* of f, denoted by $d(f)$, is the length of $b(f)$. The notions of a *k-face*, a *k^+-face* and a *k^--face* are defined analogous to the ones of a k-vertex, a k^+-vertex and a k^--vertex, respectively. Call a face *internal* if it is not the unbounded face (the unbounded face is usually denoted by f_0). Call a vertex *external* if it is on the unbounded face f_0; *internal* otherwise. An edge xy is called a $(d(x), d(y))$-edge, and x is called a $d(x)$-neighbor of y. Moreover, x is called an *isolated* neighbor of y, if xy is not incident with any cycle of length 3. For a face $f \in F$, the subgraph of G induced by $V(f)$ is denoted by $G[V(f)]$. If $u_1, u_2, \ldots, u_n$ are all vertices of $b(f)$ in a cyclic order, then we write $f = [u_1u_2 \ldots u_n]$. Two faces or cycles are *intersecting* if they have at least one vertex in common; *adjacent* if they have at least one edge in common. Let C be a cycle of G. The *length* of C, denoted $|C|$, is the number of edges of C. A *k-cycle* is a cycle of length k. A 3-cycle is usually called a *triangle*. The set of vertices inside or outside a cycle C is denoted by $int(C)$ or $ext(C)$, respectively. Consequently, $Int(C) = G - ext(C)$ and $Ext(C) = G - int(C)$ are two vertex-induced subgraphs of G. Note that the chords of C lying inside C belong to $Ext(C)$. Call a cycle C *separating* if both $int(C)$ and $ext(C)$ are not empty. Sometimes, we do not distinguish C with $V(C)$ or $E(C)$.

Let $G = (V, E, F)$ be a plane graph without 3-cycles adjacent to 3- or 5-cycles, and C a cycle of length at most 7 in G. Call C *bad* if $Int(C)$ contains a subgraph H that is isomorphic to the configuration shown in Fig. 1, where C is the boundary of the unbounded face of the subgraph H. The subgraph H is called a *bad partition* of $Int(C)$, or simply C. Call a 7^--cycle *good* if it is not bad. By the definition of a bad cycle, if a cycle C is bad then $|C| = 6$ or 7. Note that 5^--cycles are good.

A *chord* of a cycle C is an edge that connects two non-consecutive vertices of C. Let $e = xy$ be a chord of a cycle C, and P_1, P_2 the two paths of C between x and y. If the length of the cycle $C_i = P_i \cup \{e\}$ is k_i, $i = 1, 2$, then e is called a *(k_1, k_2)-chord* of C. Since G has no 3-cycles adjacent to 3- or 5-cycles, the following remark is obvious.

Remark 1. Let C be a cycle in G.
(1) If $|C| \leq 5$, then C has no chord.
(2) If $|C| = 6$, then C has at most one chord, if any, a $(4, 4)$-chord.
(3) If $|C| = 7$, then C has at most one chord, if any, a $(3, 6)$-chord or a $(4, 5)$-chord.