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# Choosability with union separation

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ABSTRACT

List coloring generalizes graph coloring by requiring the color of a vertex to be selected from a list of colors specific to that vertex. One refinement of list coloring, called *choosability with separation*, requires that the intersection of adjacent lists is sufficiently small. We introduce a new refinement, called *choosability with union separation*, where we require that the union of adjacent lists is sufficiently large. For $t \geq k$, a $(k, t)$-list assignment is a list assignment $L$ where $|L(v)| \geq k$ for all vertices $v$ and $|L(u) \cup L(v)| \geq t$ for all edges $uv$. A graph is $(k, t)$*-choosable* if there is a proper coloring for every $(k, t)$-list assignment. We explore this concept through examples of graphs that are not $(k, t)$-choosable, demonstrating sparsity conditions that imply a graph is $(k, t)$-choosable, and proving that all planar graphs are $(3, 11)$-choosable and $(4, 9)$-choosable.



## 1. Introduction

For a graph $G$ and a positive integer $k$, a $k$*-list assignment* of $G$ is a function $L$ on the vertices of $G$ such that $L(v)$ is a set of size at least $k$. An $L$*-coloring* is an assignment $c$ on the vertices of $G$ such that $c(v) \in L(v)$ for all vertices $v$ and $c(u) \neq c(v)$ for all adjacent pairs $uv$. A graph is $k$*-choosable* if there exists an $L$-coloring for every $k$-list assignment $L$ of $G$, and $G$ is $k$*-colorable* if there exists an $L$-coloring for the $k$-list assignment $L(v) = \{1, \ldots, k\}$. The minimum $k$ for which $G$ is $k$-choosable is called the *choosability* or the *list-chromatic number* of $G$ and is denoted by $\chi_\ell(G)$. Erdős, Rubin, and Taylor [6] and independently Vizing [13] introduced the concept of list coloring and demonstrated that for all $k' \geq k \geq 2$, there exist graphs that are $k$-colorable but not $k'$-choosable. Since its introduction, choosability has received significant attention and has been refined in many different ways.

One refinement of choosability is called *choosability with separation* and has received recent attention [1,4,7,8,11] since it was defined by Kratochvíl, Tuza, and Voigt [10]. Let $G$ be a graph and let $s$ be a nonnegative integer called the *separation* parameter. A $(k, k-s)$*-list assignment* is a $k$-list assignment $L$ such that $|L(u) \cap L(v)| \leq k - s$ for all adjacent pairs $uv$. We say a graph $G$ is $(k, t)$*-choosable* if, for any $(k, t)$-list assignment $L$, there exists an $L$-coloring of $G$. As the separation parameter $s$ increases, the restriction on the intersection-size of adjacent lists becomes more strict.

We introduce a complementary refinement of choosability called *choosability with union separation*. A $(k, k+s)$*-list assignment* is a $k$-list assignment $L$ such that $|L(u) \cup L(v)| \geq k + s$ for all adjacent pairs $uv$. We similarly say $G$ is $(k, t)$-choosable to imply choosability with either kind of separation, depending on whether $t \leq k$ or $k < t$. Observe that if $G$ is $(k, k+s)$-choosable, then $G$ is both $(k, k-r)$-choosable and $(k, k+r)$-choosable for all $r \geq s$. Note that if $L$ is a $(k, k-s)$-list assignment, we may assume that $|L(v)| = k$ as removing colors from lists does not violate the intersection-size requirement

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for adjacent vertices. However, when considering a $(k, k+s)$-list assignment, we may not remove colors from lists as that may violate the union-size requirement for adjacent vertices. Due to this asymmetry, we do not know if there is a function $f(k, s)$ such that every $(k, k-s)$-choosable graph is also $(k, k+f(s))$-choosable.

Thomassen [12] proved that all planar graphs are 5-choosable. The main question we consider regarding planar graphs and choosability with union separation is identifying minimum integers $t_3$ and $t_4$ such that all planar graphs are $(3, t_3)$-choosable and $(4, t_4)$-choosable. We demonstrate that $6 \le t_3 \le 11$ and $6 \le t_4 \le 9$.

Kratochvíl, Tuza, and Voigt [9] proved that all planar graphs are (4, 1)-choosable and conjecture that all planar graphs are (4, 2)-choosable. Voigt [14] constructed a planar graph that is not (4, 3)-choosable and hence is not (4, 5)-choosable. We show that $t_4 \le 9$.

**Theorem 1.** *All planar graphs are* (4, 9)*-choosable.*

A *chorded* $\ell$*-cycle* is a cycle of length $\ell$ with one additional edge. For each $\ell \in \{5, 6, 7\}$, Berikkyzy et al. [1] demonstrated that if $G$ is a planar graph that does not contain a chorded $\ell$-cycle, then $G$ is (4, 2)-choosable. The case $\ell = 4$ is notably missing from their results, especially since Borodin and Ivanova [3] proved that if $G$ is a planar graph that does not contain a chorded 4-cycle or a chorded 5-cycle, then $G$ is 4-choosable. We prove that if $G$ is a planar graph containing no chorded 4-cycle, then $G$ is (4, 7)-choosable (see Theorem 8).

Kratochvíl, Tuza, and Voigt [9] conjecture that all planar graphs are (3, 1)-choosable. Voigt [15] constructed a planar graph that is not (3, 2)-choosable and hence is not (3, 4)-choosable. In Section 2 we construct graphs that are not $(k, t)$-choosable, including a planar graph that is not (3, 5)-choosable. This hints towards a strong difference between intersection separation and union separation. We show that $t_3 \le 11$.

**Theorem 2.** *All planar graphs are* (3, 11)*-choosable.*

We also consider sparsity conditions that imply $(k, t)$-choosability. For a graph $G$, the *maximum average degree* of $G$, denoted $\mathrm{Mad}(G)$, is the maximum fraction $\frac{2|E(H)|}{|V(H)|}$ among subgraphs $H \subseteq G$. If $\mathrm{Mad}(G) < k$, then $G$ is $(k-1)$-degenerate and hence is $k$-choosable. Since $\mathrm{Mad}(K_{k+1}) = k$ and $\chi_\ell(K_{k+1}) > k$, this bound on $\mathrm{Mad}(G)$ cannot be relaxed. In Section 4, we prove that $G$ is $(k, t)$-choosable when $\mathrm{Mad}(G) < 2k - o(1)$ where $o(1)$ tends to zero as $t$ tends to infinity. This is asymptotically sharp as we construct graphs that are not $(k, t)$-choosable with $\mathrm{Mad}(G) = 2k - o(1)$.

Many of our proofs use the discharging method. For an overview of this method, see the surveys of Borodin [2], Cranston and West [5], or the overview in Berikkyzy et al. [1]. We use a very simple reducible configuration that is described by Proposition 6 in Section 3.

### *1.1. Notation*

A (simple) graph $G$ has vertex set $V(G)$ and edge set $E(G)$. Additionally, if $G$ is a plane graph, then $G$ has a face set $F(G)$. Let $n(G) = |V(G)|$ and $e(G) = |E(G)|$. For a vertex $v \in V(G)$, the set of vertices adjacent to $v$ is the *neighborhood* of $v$, denoted $N(v)$. The *degree* of $v$, denoted $d(v)$, is the number of vertices adjacent to $v$. We say $v$ is a $k$-vertex if $d(v) = k$, a $k^-$-vertex if $d(v) \le k$ and a $k^+$-vertex if $d(v) \ge k$. Let $G - v$ denote the graph given by deleting the vertex $v$ from $G$. For an edge $uv \in E(G)$, let $G - uv$ denote the graph given by deleting the edge $uv$ from $G$. For a plane graph $G$ and a face $f$, let $\ell(f)$ denote the length of the face boundary walk; say $f$ is a $k$-face if $\ell(f) = k$ and a $k^+$-face if $\ell(f) \ge k$.

## 2. Non-$(k, t)$-choosable graphs

**Proposition 3.** *For all $k \ge 2$ and $t \ge 2k - 1$, there exists a bipartite graph that is not $(k, t)$-choosable.*

**Proof.** Let $u_1, \ldots, u_k$ be nonadjacent vertices and let $L(u_1), \ldots, L(u_k)$ be disjoint sets of size $t - k + 1$. For every element $(a_1, \ldots, a_k) \in \prod_{i=1}^{k} L(u_i)$, let $A = \{a_1, \ldots, a_k\}$, create a vertex $x_A$ adjacent to $u_i$ for all $i \in [k]$, and let $L(x_A) = A$ (see Fig. 1). Notice that $|L(u_i) \cup L(x_A)| = t$ for all $i \in [k]$ and all vertices $x_A$, so $L$ is a $(k, t)$-list assignment. If there is a proper $L$-coloring $c$ of this graph, then let $A = \{c(u_i) : i \in [k]\}$; the color $c(x_A)$ is in $A$ and hence the coloring is not proper. □

Observe that the graph constructed in Proposition 3 has average degree $\frac{2k(t-k+1)^k}{k+(t-k+1)^k}$; as $t$ increases, this fraction approaches $2k$ from below. Observe that when $k = 2$ the graph built in Proposition 3 is planar, giving us the following corollary.

**Corollary 4.** *For all $t \ge 3$, there exists a bipartite planar graph that is not $(2, t)$-choosable.*

We now construct a specific planar graph that is not (3, 5)-choosable.

**Proposition 5.** *There exists a planar graph that is not* (3, 5)*-choosable.*