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Some bounds on the number of colors in interval and cyclic interval edge colorings of graphs

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ABSTRACT

An *interval t-coloring* of a multigraph G is a proper edge coloring with colors $1, \dots, t$ such that the colors of the edges incident with every vertex of G are colored by consecutive colors. A *cyclic interval t-coloring* of a multigraph G is a proper edge coloring with colors $1, \dots, t$ such that the colors of the edges incident with every vertex of G are colored by consecutive colors, under the condition that color 1 is considered as consecutive to color t. Denote by $w(G)$ ($w_c(G)$) and $W(G)$ ($W_c(G)$) the minimum and maximum number of colors in a (cyclic) interval coloring of a multigraph G, respectively. We present some new sharp bounds on $w(G)$ and $W(G)$ for multigraphs G satisfying various conditions. In particular, we show that if G is a 2-connected multigraph with an interval coloring, then $W(G) \leq 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (\Delta(G) - 1)$. We also give several results towards the general conjecture that $W_c(G) \leq |V(G)|$ for any triangle-free graph G with a cyclic interval coloring; we establish that approximate versions of this conjecture hold for several families of graphs, and we prove that the conjecture is true for graphs with maximum degree at most 4.

1. Introduction

In this paper we consider graphs which are finite, undirected, and have no loops or multiple edges, and multigraphs which may contain multiple edges but no loops. We denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a multigraph G, respectively, and by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of vertices in G, respectively. The terms and concepts that we do not define here can be found in [3,26].

An *interval t-coloring* (or *consecutive coloring*) of a multigraph G is a proper coloring of the edges by positive integers $1, \dots, t$ such that the colors of the edges incident with any vertex of G form an interval of integers, and no color class is empty. The notion of interval colorings was introduced by Asratian and Kamalian [4] (available in English as [5]), motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive time periods. We denote by $\mathfrak{N}$ the set of all interval colorable multigraphs.

All regular bipartite graphs have interval colorings, since they decompose into perfect matchings. Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper $\Delta(G)$-edge-coloring [4]. Sevastjanov [25] proved that determining whether a bipartite graph has an interval coloring is $\mathcal{NP}$-complete. Nevertheless, trees [16], regular and complete bipartite graphs [13,16,17], grids [10], and outerplanar bipartite graphs [6,11] all have interval colorings. Moreover, some families of (a, b)-biregular graphs have been proved to admit interval colorings [2,8,13,15,24,27], where a bipartite graph is (a, b)-*biregular* if all vertices in one part have degree a and all vertices in the other part have degree b.

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A *cyclic interval t-coloring* α of a multigraph G is a proper t-edge-coloring with colors $1, \ldots, t$ such that for every vertex v of G the colors of the edges incident with v either form an interval of integers or the set $\{1, \ldots, t\} \setminus \{\alpha(e) : \text{e is incident with v}\}$ is an interval of integers, and no color class is empty. This notion was introduced by de Werra and Solot [9] motivated by scheduling problems arising in flexible manufacturing systems, in particular the so-called *cylindrical open shop scheduling problem* [9]. We denote by $\mathfrak{N}_c$ the set of all cyclically interval colorable multigraphs.

Cyclic interval colorings are studied in e.g. [1,19,20,23]. In particular, the general question of determining whether a graph G has a cyclic interval coloring is $\mathcal{NP}$-complete [19] and there are concrete examples of connected graphs having no cyclic interval coloring [1,20,23]. Trivially, any multigraph with an interval coloring also has a cyclic interval coloring with $\Delta(G)$ colors, but the converse does not hold [20]. Graphs that have been proved to admit cyclic interval colorings (but not always proved to admit interval colorings) include all complete multipartite graphs [1], all Eulerian bipartite graphs of maximum degree at most 8 [1], and some families of (a, b)-biregular graphs [1,7,8].

In this paper we study upper and lower bounds on the number of colors in interval and cyclic interval colorings of various families of graphs. For a multigraph $G \in \mathfrak{N}$ ($G \in \mathfrak{N}_c$), we denote by $W(G)$ ($W_c(G)$) and $w(G)$ ($w_c(G)$) the maximum and the minimum number of colors in a (cyclic) interval coloring of G, respectively. There are some previous results on these parameters in the literature. In particular, Asratian and Kamalian [4,5] proved the fundamental result that if $G \in \mathfrak{N}$ is a triangle-free graph, then $W(G) \le |V(G)| - 1$; this upper bound is sharp for e.g. complete bipartite graphs. Kamalian [17] proved that if $G \in \mathfrak{N}$ has at least two vertices, then $W(G) \le 2|V(G)| - 3$. In [12], it was noted that this upper bound can be slightly improved if the graph has at least three vertices. Petrosyan [21] proved that these upper bounds are asymptotically sharp by showing that for any $\epsilon > 0$, there is a connected interval colorable graph G satisfying $W(G) \ge (2 - \epsilon)|V(G)|$. Kamalian [16,17] proved that the complete bipartite graph $K_{a,b}$ has an interval t-coloring if and only if $a + b - \gcd(a, b) \le t \le a + b - 1$, where $\gcd(a, b)$ is the greatest common divisor of a and b, and Petrosyan et al. [21,22] showed that the n-dimensional hypercube Q_n has an interval t-coloring if and only if $n \le t \le \frac{n(n+1)}{2}$.

For cyclic interval colorings, Petrosyan and Mkhitaryan [23] suggested the following:

Conjecture 1.1.

(i) *For any triangle-free graph* $G \in \mathfrak{N}_c$, $W_c(G) \le |V(G)|$.
(ii) *For any graph* $G \in \mathfrak{N}_c$ *with at least two vertices*, $W_c(G) \le 2|V(G)| - 3$.

If true, then these upper bounds are sharp [23].

In this paper we present some new general upper and lower bounds on the number of colors in interval and cyclic interval colorings of graphs.

For cyclic interval colorings, we prove several results related to Conjecture 1.1. In particular, we give improvements of general upper bounds on $W_c(G)$ by Petrosyan and Mkhitaryan [23], and we show that slightly weaker versions of the conjecture hold for several families of graphs. We also prove the new general upper bound that for any triangle-free graph $G \in \mathfrak{N}_c$, $W_c(G) \le \frac{\sqrt{3}+1}{2}(|V(G)| - 1)$, which is an improvement of the bound proved by Petrosyan et al. [23] for graphs with large maximum degree. These results are proved in Section 2, where we also prove that Conjecture 1.1(i) is true for graphs with maximum degree at most 4.

For interval colorings, we prove that if G is a 2-connected multigraph and $G \in \mathfrak{N}$, then $W(G) \le 1 + \left\lfloor \frac{|V(G)|}{2} \right\rfloor (\Delta(G) - 1)$. This result is proved in Section 3, where we also show that this upper bound is sharp, and give some related results. In Section 4, we obtain some lower bounds on $w(G)$ for multigraphs $G \in \mathfrak{N}$. We show that if G is an interval colorable multigraph, then $w(G) \ge \left\lceil \frac{|V(G)|}{2\alpha'(G)} \right\rceil \delta(G)$, where $\alpha'(G)$ is the size of a maximum matching in G. In particular, this implies that if G has no perfect matching and $G \in \mathfrak{N}$, then $w(G) \ge \max\{\Delta(G), 2\delta(G)\}$. Additionally, we prove that the same conclusion holds under the assumption that all vertex degrees in G are odd, $G \in \mathfrak{N}$ and $|E(G)| - \frac{|V(G)|}{2}$ is odd.

1.1. Notation and preliminary results

In this section we introduce some terminology and notation, and state some auxiliary results.

The set of neighbors of a vertex v in G is denoted by $N_G(v)$. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or just $d(v)$), the maximum degree of vertices in G by $\Delta(G)$, the minimum degree of vertices in G by $\delta(G)$, and the average degree of G by $d(G)$.

A multigraph G is *even* (*odd*) if the degree of every vertex of G is even (odd). G is *Eulerian* if it has a closed trail containing every edge of G. For two distinct vertices u and v of a multigraph G, let $E(uv)$ denote the set of all edges of G joining u with v.

The diameter of G, i.e. the greatest distance between any pair of vertices in G, is denoted by $\mathrm{diam}(G)$, and the circumference of G, i.e. the length of a longest cycle in G, is denoted by $c(G)$. We denote by $\alpha'(G)$ the size of a maximum matching in G, and by $\chi'(G)$ the chromatic index of G.

If α is a proper edge coloring of G and $v \in V(G)$, then $S_G(v, \alpha)$ (or $S(v, \alpha)$) denotes the set of colors appearing on edges incident with v. The smallest and largest colors of $S(v, \alpha)$ are denoted by $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$, respectively.

We shall use the following theorem due to Asratian and Kamalian [4,5].