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Note

A note on panchromatic colorings

Danila Cherkashin

Saint Petersburg State University, Faculty of Mathematics and Mechanics, Russian Federation
Moscow Institute of Physics and Technology, Laboratory of Advanced Combinatorics and Network Applications, Russian Federation
St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, Russian Federation

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ABSTRACT

This paper studies the quantity $p(n,r)$, that is the minimal number of edges of an n-uniform hypergraph without panchromatic coloring (it means that every edge meets every color) in r colors. If $r \leq c\frac{n}{\ln n}$ then all bounds have a type $A_1(n, \ln n, r)(\frac{r}{r-1})^n \leq p(n,r) \leq A_2(n, r, \ln r)(\frac{r}{r-1})^n$, where A_1, A_2 are some algebraic fractions. The main result is a new lower bound on $p(n,r)$ when r is at least $c\sqrt{n}$; we improve an upper bound on $p(n,r)$ if $n = o(r^{3/2})$.

Also we show that $p(n,r)$ has upper and lower bounds depending only on n/r when the ratio n/r is small, which cannot be reached by the previous probabilistic machinery.

Finally we construct an explicit example of a hypergraph without panchromatic coloring and with $(\frac{r}{r-1} + o(1))^n$ edges for $r = o(\sqrt{\frac{n}{\ln n}})$.

1. Introduction

A hypergraph is a pair (V, E), where V is a finite set whose elements are called vertices and E is a family of subsets of V, called edges. A hypergraph is n-uniform if every edge has size n. A vertex r-coloring of a hypergraph (V, E) is a map $c : V \to \{1, \dots, r\}$.

An r-coloring of vertices of a hypergraph is called *panchromatic* if every edge contains a vertex of every color. The problem of the existence of a panchromatic coloring of a hypergraph was stated in the local form by P. Erdős and L. Lovász in [4]. They proved that if every edge of an n-uniform hypergraph intersects at most $r^{n-1}/4(r-1)^n$ other edges then the hypergraph has a panchromatic r-coloring. Then A. Kostochka in [7] stated the problem in the present form and linked it with the r-choosability problem using ideas by P. Erdős, A.L. Rubin and H. Taylor from [5]. Also A. Kostochka and D.R. Woodall [9] found some sufficient conditions on a hypergraph to have a panchromatic coloring in terms of Hall ratio. Reader can find a survey on history and results on the related problems in [8,11].

1.1. Upper bounds

Using the results from [1] A. Kostochka proved [7] that for some constants $c_1, c_2 > 0$

$$\frac{1}{r}e^{c_1\frac{n}{r}} \leq p(n,r) \leq re^{c_2\frac{n}{r}}. \tag{1}$$

In works [13,14] D. Shabanov gives the following upper bounds:

$$p(n,r) \leq c\frac{n^2 \ln r}{r^2}\left(\frac{r}{r-1}\right)^n, \text{ if } 3 \leq r = o(\sqrt{n}), n > n_0;$$

E-mail address: matelk@mail.ru.

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$$p(n,r) \le c\frac{n^{3/2}\ln r}{r}\left(\frac{r}{r-1}\right)^n, \text{ if } r = O(n^{2/3}) \text{ and } n_0 < n = O(r^2); \tag{2}$$

$$p(n,r) \le c\max\left(\frac{n^2}{r}, n^{3/2}\right)\ln r\left(\frac{r}{r-1}\right)^n \text{ for all } n, r \ge 2.$$

Let us introduce the quantity $p'(n,r)$ that is the minimal number of edges in an n-uniform hypergraph $H = (V, E)$ such that any subset of vertices $V' \subset V$ with $|V'| \ge \left\lceil \frac{r-1}{r}|V| \right\rceil$ contains an edge.

Note that by pigeonhole principle every vertex r-coloring contains a color of size at most $\left\lfloor \frac{1}{r}|V| \right\rfloor$. So the complement to this color has size at least $|V| - \left\lfloor \frac{1}{r}|V| \right\rfloor = \left\lceil \frac{r-1}{r}|V| \right\rceil$. Hence, $p(n,r) \le p'(n,r)$. This argument is in the spirit of the standard estimation of the chromatic number via the independence number.

The following theorem gives better upper bound in the case when $n = o(r^{3/2})$.

Theorem 1.1. *The following inequality holds for every $n \ge 2$, $r \ge 2$*

$$p'(n,r) \le c\frac{n^2\ln r}{r}\left(\frac{r}{r-1}\right)^n.$$

It immediately implies

$$p(n,r) \le c\frac{n^2\ln r}{r}\left(\frac{r}{r-1}\right)^n.$$

1.2. Lower bounds

We start by noting that an evident probabilistic argument gives $p(n,r) \ge \frac{1}{r}(\frac{r}{r-1})^n$. This gives lower bound (1) with $c_1 = 1$. This was essentially improved by D. Shabanov in [13]:

$$p(n,r) \ge c\frac{1}{r^2}\left(\frac{n}{\ln n}\right)^{1/3}\left(\frac{r}{r-1}\right)^n \text{ for } n, r \ge 2, r < n.$$

Next, A. Rozovskaya and D. Shabanov [12] showed that

$$p(n,r) \ge c\frac{1}{r^2}\sqrt{\frac{n}{\ln n}}\left(\frac{r}{r-1}\right)^n \text{ for } n, r \ge 2, r \le \frac{n}{2\ln n}.$$

Using the Alterations method (see Section 3 of [2]) we can get the following lower bound for all the range of n, r. It gives better results when $r \ge c\sqrt{n}$.

Theorem 1.2. *For $n \ge r \ge 2$ holds*

$$p(n,r) \ge e^{-1}\frac{r-1}{n-1}e^{\frac{n-1}{r-1}}.$$

There is a completely different way to get almost the same bound. First, we need to prove intermediate bound. It is based on the geometric rethinking of A. Pluhár's ideas [10].

Theorem 1.3. *For $n \ge r \ge 2$ such that $r \le c\frac{n}{\ln n}$ holds*

$$p(n,r) \ge c\max\left(\frac{n^{1/4}}{r\sqrt{r}}, \frac{1}{\sqrt{n}}\right)\left(\frac{r}{r-1}\right)^n.$$

Combining Theorems 1.2 and 1.3 we prove the following theorem.

Theorem 1.4. *For $n \ge r \ge 2$ such that $\sqrt{n} \le r \le c'\frac{n}{\ln n}$ holds*

$$p(n,r) \ge c\frac{r}{n}e^{\frac{n}{r}}.$$

Remark 1.5. Theorem 1.3, unlike Theorems 1.2 and 1.4, admits a local version.