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Note

A short disproof of Euler's conjecture based on quasi-difference matrices and difference matrices[☆]

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ABSTRACT

In this note, two classes of quasi-difference matrices, $(2n+2, 4; 1, 1; n)$-QDM and $(4n+1, 4; 1, 1; 2n-1)$-QDM, are constructed. Combining the known results of quasi-difference matrices and difference matrices, a new short disproof of Euler's conjecture on mutually orthogonal Latin squares is given.

1. Introduction

A Latin square of side n (or order n) is an $n \times n$ array in which each cell contains a single symbol from an n-set, such that each symbol occurs exactly once in each row and exactly once in each column. Two Latin squares with the same order are orthogonal if each symbol of one square meets each symbol of another exactly once when one square is superimposed on another. A set of Latin squares is mutually orthogonal, or a set of MOLS, if every pair of distinct squares is orthogonal. $N(n)$ is the maximal number of Latin squares in a set of MOLS of side n.

In 1779, Euler studied the thirty-six officers problem: arranging thirty-six officers, drawn from six different ranks and six different regiments (one of each rank from each regiment), into a square such that in each line (both horizontal and vertical) there are six officers of different ranks and different regiments. Euler conjectured that such an $n \times n$ array does not exist for $n = 6$, nor does one exist whenever $n \equiv 2 \pmod 4$. This was known as Euler's conjecture. The equivalent object of such an array is two MOLS of side n. Euler's conjecture can be stated as $N(n) = 1$ whenever $n \equiv 2 \pmod 4$.

Euler's conjecture is true for $n = 2$. In 1900, Tarry [8] proved that $N(6) = 1$ undertook a lengthy case analysis. In 1960, Bose et al. [1] showed that $N(n) \geq 2$ for all $n \equiv 2 \pmod 4$ and $n \geq 10$ by means of PBD construction and method of differences. In 1960, Sade [6] showed that $N(n) \geq 2$ for all $n \equiv 2 \pmod 4$ and $n \geq 482$ by means of his singular direct product of quasigroups. In 1974, Wilson [9] showed that $N(n) \geq 2$ for all $n \equiv 2 \pmod 4$ and $n \geq 18$ by means of Wilson's fundamental construction. In 1977, Zhu [10] showed that $N(n) \geq 2$ for all $n \equiv 2 \pmod 4$ and $n \geq 10$ by means of sum composition method. In 1984, Stinson [7] gave a short proof of $N(6) = 1$. In 1994, Dougherty [4] gave another proof of $N(6) = 1$ using the tools of algebraic coding theory. For more details about Euler's conjecture and results about $N(n)$, the interested reader may refer to the detailed survey [2] and references therein.

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In this note, the existence of two classes of quasi-difference matrices, $(2n+2, 4; 1, 1; n)$-QDM and $(4n+1, 4; 1, 1; 2n-1)$-QDM, are confirmed. Combining the known results of quasi-difference matrices and difference matrices, a new short disproof of Euler's conjecture on mutually orthogonal Latin squares is given.

2. Preliminary

In this section, we present some notations, terminologies, and known results in design theory at first, and then introduce the construction of MOLS based on quasi-difference matrix and difference matrix.

A transversal design of order (or group size) n, block size k, and index λ, denoted $TD_\lambda(k, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a set of elements, $\mathcal{G}$ is a partition of V into k classes (the groups), each of size n, $\mathcal{B}$ is a collection of subsets of V (called blocks) such that no two points in the same group appear in any block, and any two other points appear in exactly λ blocks. When $\lambda = 1$, one writes simply $TD(k, n)$.

Lemma 2.1 ([3]). *The existence of k MOLS of side n is equivalent to the existence of a transversal design of index one, block size $k+2$, and group size n, namely, a $TD(k+2, n)$.*

A $TD_\lambda(k, n)$, $(V, \mathcal{G}, \mathcal{B})$, is α-resolvable if its blocks can be partitioned into set $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s$, where every element of V occurs exactly α times in each $\mathcal{B}_i$. The classes $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s$ are α-parallel classes. When $\alpha = 1$, the design is completely resolvable, denoted $RTD_\lambda(k, n)$, and the classes are parallel classes. An $RTD_\lambda(k, n)$ is a 1-resolvable $TD_\lambda(k, n)$.

Lemma 2.2 ([3]). *The existence of a $TD(k+1, n)$ is equivalent to the existence of an $RTD(k, n)$.*

Let $(G, \odot)$ be a group of order g. A $(g, k; \lambda)$-difference matrix is a $k \times g\lambda$ matrix $D = (d_{ij})$ with entries from G, so that for each $1 \le i < j \le k$, the multiset $\{d_{il} \odot d_{jl}^{-} : 1 \le l \le g\lambda\}$ (the difference list) contains every element of G exactly λ times. When G is abelian, typically additive notation is used, so that differences $d_{il} - d_{jl}$ are employed.

Lemma 2.3 ([3]). *If a $(g, k; 1)$-difference matrix exists, then an $RTD(k, n)$ exists, hence a $TD(k+1, n)$ exists.*

In 2005, Ge obtained the following result.

Lemma 2.4 ([5]). *A $(g, 4; 1)$-difference matrix exists if and only if $g \ge 4$ and $g \not\equiv 2 \pmod 4$.*

An incomplete transversal design of order (or group size) n, block size k, index λ, and hole sizes $b_1, b_2, \ldots, b_s$, denoted $ITD_\lambda(k, n; b_1, b_2, \ldots, b_s)$, is a quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where V is a set of kn elements, $\mathcal{G}$ is a partition of V into k classes (the groups), each of size n, $\mathcal{H}$ is a set of disjoint subsets $H_1, H_2, \ldots, H_s$ of V (the holes), $\mathcal{B}$ is a collection of k-subsets of V (the blocks), with the properties that:

(1) for each $1 \le i \le s$ and each $G \in \mathcal{G}$, $|G \cap H_i| = b_i$;
(2) every unordered pair of elements from the same hole is contained in no blocks;
(3) every unordered pair of elements from the same group is contained in no blocks; and
(4) every unordered pair of elements from neither the same hole nor the same group is contained in λ blocks.

When $\lambda = 1$, one writes simply $ITD(k, n; b_1, b_2, \ldots, b_s)$. A set of blocks is an α-partial parallel class if each element of V not contained in any holes occurs exactly α times in the set.

Lemma 2.5 ([3]). *If both an $ITD(k, n; b)$ and a $TD(k, b)$ exists, then a $TD(k, n)$ exists.*

Let G be an abelian group of order n. An $(n, k; \lambda, \mu; u)$-quasi-difference matrix (QDM) is a matrix $Q = (q_{ij})$ with k rows and $\lambda(n-1+2u)+\mu$ columns, with each entry either empty (usually denoted by $-$) or containing a single element of G. Each row contains exactly λu empty entries, and each column contains at most one empty entry. Furthermore, for each $1 \le i < j \le k$, the multiset $\{q_{il} - q_{jl} : 1 \le l \le \lambda(n-1+2u)+\mu$, with q_{il} and q_{jl} not empty$\}$ contains every nonzero element of G exactly λ times and contains 0 exactly μ times.

Lemma 2.6 ([3]). *If an $(n, k; 1, 1; u)$-quasi-difference matrix exists, then so does an $ITD(k, n+u; u)$ with $n-(k-2)u$ partial parallel classes exists.*

Combining Lemmas 2.3, 2.6, and 2.5, we have the following.

Lemma 2.7. *If both an $(n, k; 1, 1; u)$-quasi-difference matrix and a $(u, 4; 1)$-difference matrix exist, then so does a $TD(k, n+u)$ with v parallel partial classes where $v = \min\{n-(k-2)u, u\}$.*

The next lemma was obtained by Bose et al. [1]. By taking $n = 2t+1$ in this lemma, they were able to obtain an $(8t+7, 4, 1, 1; 4t+3)$-QDM (and hence also 2 MOLS$(12t+10)$) for all integers $t \ge 0$.

Lemma 2.8 ([1]). *A $(4n+3, 4; 1, 1; 2n+1)$-QDM exists for each positive integer n.*

To get our main result, we shall construct two classes of quasi-difference matrices in Section 3 and Section 4, respectively.