Discrete Mathematics 341 (2018) 658–664

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Proper colouring Painter–Builder game

Małgorzata Bednarska-Bzdęga [a], Michael Krivelevich [b], Viola Mészáros [c], Clément Requilé [d,*]

[a] *Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Poland*
[b] *School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 6997801, Israel*
[c] *Bolyai Intézet, Aradi vértanúk tere 1, Szeged 6720, Hungary*
[d] *Institute for Algebra, Johannes Kepler Universität Linz, Austria*

ARTICLE INFO

Article history:
Received 17 September 2017
Received in revised form 9 November 2017
Accepted 10 November 2017

Keywords:
Combinatorial games
Graph colouring

ABSTRACT

We consider the following two-player game, parametrised by positive integers n and k. The game is played between Painter and Builder, alternately taking turns, with Painter moving first. The game starts with the empty graph on n vertices. In each round Painter colours a vertex of her choice by one of the k colours and Builder adds an edge between two previously unconnected vertices. Both players must adhere to the restriction that the game graph is properly k-coloured. The game ends if either all n vertices have been coloured, or Painter has no legal move. In the former case, Painter wins the game; in the latter one, Builder is the winner. We prove that the minimal number of colours $k = k(n)$ allowing Painter's win is of logarithmic order in the number of vertices n. Biased versions of the game are also considered.

1. Introduction

We study the following perfect information game. Let $n \geqslant 2$ and k be natural numbers. The Painter–Builder k-colouring game on $[n] = \{1, 2, \ldots, n\}$ is played between Painter (who starts the game) and Builder. The players take turns to create/grow a graph G, the *game graph*, with a k-colouring of its vertices in the following way. The vertex set of G is $[n]$ and at the beginning there are no edges. In every round Painter colours a previously uncoloured vertex with one of k colours. Then Builder adds to G an edge between two previously unconnected vertices. Throughout the game, both players must adhere to the restriction that the game graph is properly k-coloured. The game ends if either all vertices are coloured or Painter has no legal move. In the first case, Painter wins the game (and the game lasts n rounds), and in the latter one Painter is the loser. Observe that Painter loses the game if and only if at some point of the game there is an uncoloured vertex v, having neighbours in each one of the k colours.

The game belongs to a wide class of Builder–Painter games, in which Builder reveals edges or vertices of a graph (which is the board of the game) and Painter colours edges or vertices of the board. The aim of Painter is to avoid building a monochromatic copy of a graph from a given graph family. There are different variants of the game, but they are mostly linked to the so called online size Ramsey number. The online size Ramsey number has been investigated in numerous papers starting with [1,2] by Beck. Builder selects one edge per turn, then Painter has to colour it with one of k colours. Builder is trying to force Painter to complete a monochromatic copy of some fixed graph H in as few rounds as possible on

* Corresponding author.
E-mail addresses: mbed@amu.edu.pl (M. Bednarska-Bzdęga), krivelev@post.tau.ac.il (M. Krivelevich), viola@math.u-szeged.hu (V. Mészáros), clement.requile@jku.at (C. Requilé).

https://doi.org/10.1016/j.disc.2017.11.008
0012-365X/

an unbounded vertex set. The *online size Ramsey number* of H is the minimum number of rounds until Builder achieves his goal assuming both players play optimally. Most of the results were obtained for the case of $k = 2$ colours. In particular, Conlon [4] proved that for infinitely many values of l the online size Ramsey number of K_l is exponentially smaller than the size Ramsey number of K_l. Grytczuk, Kierstead and Prałat [6] investigated the online size Ramsey number for paths and for some other graphs. Grytczuk, Hałuszczak and Kierstead [5] described yet another variant of the game: Builder selects an edge and Painter colours it but there is a restriction. While Builder tries to force Painter to build some fixed monochromatic graph H, Builder cannot choose edges arbitrarily. He has to keep the constructed graph in a prescribed class of finite graphs $\mathcal{G}$ throughout the game. The task is to determine the winner for a given pair $H, \mathcal{G}$. They showed that for the case of two colours, Builder has a winning strategy for any k-colourable graph H in the class of k-colourable graphs. Belfrage, Mütze and Spöhel [3] used Builder–Painter games as a tool for studying a probabilistic one-player online Ramsey game. In their version, Builder reveals one edge per turn and he has to abide by the restriction that the edge density of the already revealed graph must not exceed a given bound d.

Let us stress that in all above versions of the Builder–Painter game, Painter colours edges and not vertices. However, the vertex colouring games have also been studied. Motivated by a problem of an online colouring random process, Mütze, Rast and Spöhel [7] and Mütze and Spöhel [8] considered the following game. Given $k \in \mathbb{N}$, $d > 0$, a graph H and an infinite set of vertices, in each round Builder draws edges connecting a previously unselected vertex v to some of the previously selected vertices, and Painter has to respond by colouring v with one of k colours. Builder is not allowed to draw an edge that would create a subgraph with edge density greater than d. Builder's aim is to force Painter to create a monochromatic copy of H.

The above Builder–Painter game has a different focus from the k-colouring game in the present paper. One important characteristic of our game is that it only considers proper colourings. Moreover, subject to the constraint that the graph be properly coloured, Builder can draw any edge and Painter can colour any uncoloured vertex, even one not incident to any edge selected by Builder. The constraint of keeping the edge density low is voided, however, we will consider a version of the Painter–Builder k-colouring game with the constraint of keeping the graph created by Builder 2-colourable. We will show that, somewhat surprisingly, even then Builder can force Painter to use many colours. More precisely, Painter needs $\Theta(\log n)$ colours to colour all n vertices properly.

Let $k_{\min}(n)$ denote the minimum number of colours such that Painter has a winning strategy in the k-colouring game on $[n]$. Clearly $k_{\min}(n) \leqslant n$ since Painter trivially wins the n-colouring game on $[n]$.

Our main result asserts that $k_{\min}(n) = \Theta(\log n)$.

Theorem 1. *For every $n > 10^8$,*

$$0.01\log_2 n < k_{\min}(n) \leqslant \log_2 n + 1.$$

Moreover, if $k \leqslant 0.01\log_2 n$, then Builder has a strategy to win the k-colouring game on $[n]$, while keeping the game graph bipartite all game long.

Let us emphasise that we made no effort to optimise the constants in the theorem. The proof of the theorem will be discussed in the following two sections. We present a random argument for the existence of a winning strategy for Painter in Section 2, and we exhibit a winning strategy for Builder in Section 3. Section 4 is devoted to the biased version of the Painter–Builder k-colouring game, where Builder claims $b \geqslant 1$ edges each time. At every moment of the game by the *game graph* we mean the graph with the vertex set $[n]$ and the edge set consisting of all edges drawn by Builder so far.

2. Painter's strategy

In this section, we describe a greedy-random strategy for Painter for the $(\lfloor \log_2 n \rfloor + 1)$-colouring game on $[n]$. We prove that with this strategy she wins with positive probability against any strategy of Builder. Suppose the colours are the integers $1, 2, 3, \ldots, \lfloor \log_2 n \rfloor + 1$. The strategy of Painter is as follows.

She starts by colouring a vertex with colour 1. Then, every time Builder adds a new edge $e = uv$ to the graph, Painter responds by selecting a vertex to colour in the following way:

- If both endpoints u, v are uncoloured, then Painter chooses one of u, v at random, with probability 1/2.
- If exactly one of u, v is uncoloured, then Painter selects it.
- If both endpoints are coloured, then Painter chooses an arbitrary uncoloured vertex.

After having chosen which vertex to colour, Painter colours it with the smallest available colour (that is with the smallest colour currently not assigned to its neighbours). If there is no such colour among all colours available for the game, then Painter forfeits the game.

To analyse this strategy, let us denote by A_v the (random) event that at some point in the game, a vertex $v \in [n]$ is still uncoloured, and has degree $\lfloor \log_2 n \rfloor + 1$. The probability of such an event is bounded as follows:

$$\mathbb{P}(A_v) \leq \left(\frac{1}{2}\right)^{\lfloor \log_2 n \rfloor + 1} < \frac{1}{n}.$$