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Note

Connectivity keeping stars or double-stars in 2-connected graphs[☆]

Yingzhi Tian [a,*], Jixiang Meng [a], Hong-Jian Lai [b], Liqiong Xu [c]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, PR China
[b] Department of Mathematics, West Virginia University, Morgantown, WV 26506, USA
[c] School of Science, Jimei University, Xiamen, Fujian 361021, PR China

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ABSTRACT

In Mader (2010), Mader conjectured that for every positive integer k and every finite tree T with order m, every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ contains a subtree T' isomorphic to T such that $G - V(T')$ is k-connected. In the same paper, Mader proved that the conjecture is true when T is a path. Diwan and Tholiya (2009) verified the conjecture when $k = 1$. In this paper, we will prove that Mader's conjecture is true when T is a star or double-star and $k = 2$.

1. Introduction

In this paper, *graph* always means a finite, undirected graph without multiple edges and without loops. For graph-theoretical terminologies and notation not defined here, we follow [1]. For a graph G, the vertex set, the edge set, the minimum degree and the connectivity number of G are denoted by $V(G)$, $E(G)$, $\delta(G)$ and $\kappa(G)$, respectively. The *order* of a graph G is the cardinality of its vertex set, denoted by $|G|$. k and m always denote positive integers.

In 1972, Chartrand, Kaugars, and Lick proved the following well-known result.

Theorem 1.1 ([2]). *Every k-connected graph G of minimum degree $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor$ has a vertex u with $\kappa(G-u) \geq k$.*

Fujita and Kawarabayashi proved in [4] that every k-connected graph G with minimum degree at least $\lfloor \frac{3}{2}k \rfloor + 2$ has an edge $e = uv$ such that $G - \{u, v\}$ is still k-connected. They conjectured that there are similar results for the existence of connected subgraphs of prescribed order $m \geq 3$ keeping the connectivity.

Conjecture 1 ([4]). *For all positive integers k, m, there is a (least) non-negative integer $f_k(m)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + f_k(m)$ contains a connected subgraph W of exact order m such that $G - V(W)$ is still k-connected.*

They also gave examples in [4] showing that $f_k(m)$ must be at least m for all positive integers k, m. In [5], Mader proved that $f_k(m)$ exists and $f_k(m) = m$ holds for all k, m.

Theorem 1.2 ([5]). *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor + m - 1$ for positive integers k, m contains a path P of order m such that $G - V(P)$ remains k-connected.*

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* Corresponding author.
E-mail addresses: tianyzhxj@163.com (Y. Tian), mjx@xju.edu.cn (J. Meng), hjlai@math.wvu.edu (H.-J. Lai), 200661000016@jmu.edu.cn (L. Xu).

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In the same paper, Mader [5] asked whether the result is true for any other tree T instead of a path, and gave the following conjecture.

Conjecture 2 (*[5]*). *For every positive integer k and every finite tree T, there is a least non-negative integer $t_k(T)$, such that every k-connected, finite graph G with $\delta(G) \geq \lfloor \frac{3}{2}k \rfloor - 1 + t_k(T)$ contains a subgraph $T' \cong T$ with $\kappa(G - V(T')) \geq k$.*

Mader showed that $t_k(T)$ exists in [6].

Theorem 1.3 (*[6]*). *Let G be a k-connected graph with $\delta(G) \geq 2(k-1+m)^2 + m - 1$ and let T be a tree of order m for positive integers k, m. Then there is a tree $T' \subseteq G$ isomorphic to T such that $G - V(T')$ remains k-connected.*

Mader further conjectured that $t_k(T) = |T|$.

Conjecture 3 (*[5]*). *For every positive integer k and every tree T, $t_k(T) = |T|$ holds.*

Theorem 1.2 showed that Conjecture 3 is true when T is a path. Diwan and Tholiya [3] proved that the conjecture holds when $k = 1$. In the next section, we will verify that Conjecture 3 is true when T is a star and $k = 2$. It is proved in the last section that Conjecture 3 is true when T is a double-star and $k = 2$.

A *block* of a graph G is a maximal connected subgraph of G that has no cut vertex. Note that any block of a connected graph of order at least two is 2-connected or isomorphic to K_2.

For a vertex subset U of a graph G, $G[U]$ denotes the subgraph induced by U and $G-U$ is the subgraph induced by $V(G)-U$. The *neighborhood* $N_G(U)$ of U is the set of vertices in $V(G) - U$ which are adjacent to some vertex in U. If $U = \{u\}$, we also use $G - u$ and $N_G(u)$ for $G - \{u\}$ and $N_G(\{u\})$, respectively. The *degree* $d_G(u)$ of u is $|N_G(u)|$. If H is a subgraph of G, we often use H for $V(H)$. For example, $N_G(H)$, $H \cap G$ and $H \cap U$ mean $N_G(V(H))$, $V(H) \cap V(G)$ and $V(H) \cap U$, respectively. If there is no confusion, we always delete the subscript, for example, $d(u)$ for $d_G(u)$, $N(u)$ for $N_G(u)$, $N(U)$ for $N_G(U)$ and so on. A *tree* is a connected graph without cycles. A *star* is a tree that has exact one vertex with degree greater than one. A *double-star* is a tree that has exact two vertices with degree greater than one.

2. Connectivity keeping stars in 2-connected graphs

Theorem 2.1. *Let G be a 2-connected graph with minimum degree $\delta(G) \geq m + 2$, where m is a positive integer. Then for a star T with order m, G contains a star T' isomorphic to T such that $G - V(T')$ is 2-connected.*

Proof. If $m \leq 3$, then T is a path, and the theorem holds by Theorem 1.2. Thus we assume $m \geq 4$ in the following.

Since $\delta(G) \geq m+2$, there is a star $T' \subseteq G$ with $T' \cong T$. Assume $V(T') = \{u, v_1, \ldots, v_{m-1}\}$ and $E(T') = \{uv_i | 1 \leq i \leq m-1\}$. We say T' is a star rooted at u or with root u. Let $G' = G - T'$. Let B be a maximum block in G' and let l be the number of components of $G' - B$. If $l = 0$, then $B = G'$ is 2-connected. So we may assume that $l \geq 1$. Let $H_1, \ldots, H_l$ be the components of $G' - B$ with $|H_1| \geq \cdots \geq |H_l|$.

Take such a star T' so that

(P1) $|B|$ is as large as possible,

(P2) $(|H_1|, \ldots, |H_l|)$ is as large as possible in lexicographic order, subject to (P1).

We will complete the proof by a series of claims.

Claim 1. *$|N(H_i) \cap B| \leq 1$ and $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.*

Since B is a block of G', we have $|N(H_i) \cap B| \leq 1$ for each $i \in \{1, \ldots, l\}$. Since G is 2-connected, $|N(H_i) \cap V(T')| \geq 1$ for each $i \in \{1, \ldots, l\}$.

Claim 2. *$l = 1$.*

Assume $l \geq 2$. By Claim 1, there is an edge th between T' and H_1, where $t \in T'$ and $h \in H_1$. Choose a vertex $x \in H_l$. Since $\delta(G) \geq m + 2$ and $|N(H_l) \cap B| \leq 1$ (by Claim 1), we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But then either there is a larger block than B in $G - T''$, or $G - T'' - B$ contains a larger component than H_1 ($H_1 \cup \{t\}$ is contained in a component of $G - T'' - B$), which contradicts to (P1) or (P2).

Claim 3. *$|N(t) \cap B| \leq 1$ and $|N(t) \cap H_1| \geq 2$ for any vertex $t \in V(T')$.*

Assume $|N(t) \cap B| \geq 2$. Choose a vertex $x \in H_1$. Since $\delta(G) \geq m + 2$ and $|N(H_1) \cap B| \leq 1$, we have $|N(x) \setminus (B \cup \{t\})| \geq m + 2 - 1 - 1 = m$. Thus we can choose a star $T'' \cong T$ with root x such that $V(T'') \cap (B \cup \{t\}) = \emptyset$. But $G - T''$ has a block containing $B \cup \{t\}$ as a subset, which contradicts to (P1). Thus $|N(t) \cap B| \leq 1$ holds. By $d(t) \geq m + 2$ and $|N(t) \cap B| \leq 1$, we have $|N(t) \cap H_1| = d(t) - |N(t) \cap B| - |N(t) \cap T'| \geq m + 2 - 1 - (m - 1) = 2$.

Claim 4. *For any edge $t_1t_2 \in E(T')$, $|N(\{t_1, t_2\}) \cap B| \leq 1$ holds.*