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Note

Nonexistence of certain pseudogeometric graphs

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ABSTRACT

Izo(r) is the pseudogeometric graph for $\mathrm{pG}_r(s, t)$ which satisfies the Krein equality and of which the local graph also satisfies the Krein equality. We prove that Izo(r) exists if and only if a spherical tight 4-design on $S^{(2r+1)^2-4}$ exists. Nonexistence of infinitely many spherical tight 4-designs is well known. It gives the nonexistence of Izo(r) for infinitely many r. Especially, Izo(4) does not exist, which was the first open case.

1. Introduction

An incidence system (X, L), where X is a set of points and L is a set of lines, is called an *α-partial geometry of order (s, t)* if the following conditions hold.

(1) Two lines intersect at most one point.
(2) Each line contains $(s + 1)$ points.
(3) Each point lies on $(t + 1)$ lines.
(4) For each point a and line L such that $a \notin L$, there are exactly α lines which contain a and intersect L.

We denote it by $\mathrm{pG}_\alpha(s, t)$. It is well-known that the point graph of a $\mathrm{pG}_\alpha(s, t)$ whose vertices are the points of the geometry and two distinct vertices are adjacent if they are collinear is a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = ((s + 1)(1 + \frac{st}{\alpha}), s(t + 1), (s - 1) + (\alpha - 1)t, \alpha(t + 1)).$$

A strongly regular graph with above parameters is called a *pseudogeometric graph for $\mathrm{pG}_\alpha(s, t)$*. A pseudogeometric graph for $\mathrm{pG}_\alpha(s, t)$ is called *geometric* if it is a point graph of a partial geometry.

Cameron, Goethals, and Seidel [10] proved that every pseudogeometric graph for $\mathrm{pG}_1(s, s^2)$ is geometric for all s. They observed that the pseudogeometric graph for $\mathrm{pG}_1(s, s^2)$ satisfies equality in the Krein condition. The reader is referred to [10] for the exact formulation of the Krein condition. Based on this observation, they posed the following problem.

Problem 1.1. Is a pseudogeometric graph for $\mathrm{pG}_\alpha(s, t)$ satisfying equality in the Krein condition geometric?

This problem has been open for almost 40 years, and recently Östergård and Soicher [17] gave the first counterexample for this problem.

Also there are some results of this problem.

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Theorem 1.2 ([13]). *Suppose that a pseudogeometric graph G for $pG_\alpha(s,t)$ satisfies the Krein equality and a is any vertex in G. Let $\Gamma(a)$ be the subgraph induced on the set of all vertices of the graph G adjacent to a. Then the following are true.*

(1) *$\Gamma(a)$ is pseudogeometric for $pG_{\alpha-1}(s-1,x)$, where $x=\frac{(\alpha-1)t}{s-1}$.*
(2) *If $\Gamma(a)$ also satisfies the Krein equality, then $s=2\alpha$.*

A pseudogeometric graph which satisfies the condition (2) in the above theorem with $\alpha = r$ is denoted by *Izo(r)*. The definition of Izo(r) can be rewritten in terms of *locally $\mathcal{F}$-graphs*.

Definition 1.3. Let $\mathcal{F}$ be a family of graphs. A graph G is called *locally $\mathcal{F}$* if $\Gamma(a)$ belongs to $\mathcal{F}$ for every $a \in V(G)$.

Definition 1.4. Izo(r) is a pseudogeometric graph G for $pG_r(s,t)$ such that

(1) G is a locally $\mathcal{F}_1$-graph, where $\mathcal{F}_1$ is the family of pseudogeometric graphs for $pG_{r-1}(s-1,x)$, where $x=\frac{(r-1)t}{s-1}$.
(2) For all a in $V(G)$, the local graph $\Gamma(a)$ is a locally $\mathcal{F}_2$-graph, where $\mathcal{F}_2$ is the family of pseudogeometric graphs for $pG_{r-2}(s-2,\frac{(r-2)x}{s-2})$.

Izo(1) is the point graph of the generalized quadrangle of order (2,4). Izo(2) is the McLaughlin graph. Those two graphs are only known Izo(r). Mahknev [14] proved that there is no Izo(3). Makhnev [15] gave some partial results about the existence of Izo(4). In this paper, we give a new result about the nonexistence of Izo(r). This is the main theorem of this paper.

Theorem 1.5. *Izo(r) exists if and only if a spherical tight 4-design exists on $S^{(2r+1)^2-4}$.*

Corollary 1.6. *Izo(r) does not exist for r such that*

(1) *$r=2k$ for some $k \not\equiv 1$ (mod 3).*
(2) *Both k and $2k+1$ are square free.*

By above corollary, Izo(r) does not exist for $r = 4, 6, 10, 12, 22, 28, 30, 34, 42, 46\cdots$. This is the first result about the nonexistence of Izo(r) for $r \geq 4$. Especially, this result contains the nonexistence of Izo(4), which was the first unknown case.

2. Spherical designs

The proof of the Theorem 1.5 is based on the basic theory of spherical designs. In this section, we discuss about spherical designs briefly.

2.1. Definition of spherical designs

Spherical designs were defined by Delsarte, Goethals, and Seidel [12] in 1977. We consider a finite subset X on the unit sphere S^d in $(d+1)$-dimensional Euclidean space $\mathbb{R}^{d+1}$.

Definition 2.1. A finite subset X of unit sphere S^d is called a *spherical t-design* if

$$\frac{1}{|S^d|}\int_{S^d} f(x)dS(x) = \frac{1}{|X|}\sum_{x\in X} f(x)$$

holds for any polynomial $f(x)$ of degree at most t, with the usual integral on the unit sphere.

We say that X has the *strength t*, if X is a spherical t-design and not a spherical $(t+1)$-design. There are some equivalent conditions to the definition of the spherical t-design.

Theorem 2.2 ([1]). *The following are equivalent.*

(1) *X is a spherical t-design on S^d.*
(2) *$\sum_{x\in X} f(x)=0$ for any homogeneous harmonic polynomial f with $1 \leq \deg f \leq t$.*
(3) *$\sum_{x\in X} f(x)=\sum_{x\in X} f(\sigma(x))$, for any polynomial f with $\deg f \leq t$ and for any orthogonal transformation $\sigma \in O(d)$.*

2.2. Association schemes and spherical designs

Let X be a spherical t-design on S^d. Let $A(X)=\{u\cdot v | u, v \in X, u \neq v\}$. We say that X is an s-distance set if $|A(X)| = s$. Delsarte, Goethals, and Seidel [12] proved that a spherical design has a structure of a Q-polynomial association scheme under some conditions. The definition of association schemes is as follows.