Discrete Mathematics 341 (2018) 1131–1135

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

No acute tetrahedron is an 8-reptile

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ARTICLE INFO

Article history:
Received 9 March 2017
Received in revised form 3 August 2017
Accepted 11 October 2017
Available online 10 November 2017

Keywords:
Tessellation
Tetrahedron
Reptile

ABSTRACT

An r-gentiling is a dissection of a shape into $r \geq 2$ parts which are all similar to the original shape. An r-reptiling is an r-gentiling of which all parts are mutually congruent. The complete characterization of all reptile tetrahedra has been a long-standing open problem. This note concerns acute tetrahedra in particular. We find that no acute tetrahedron is an r-gentile or r-reptile for any $r < 10$. The proof is based on showing that no acute spherical diangle can be dissected into less than ten acute spherical triangles.

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1. Introduction

Let T be a closed set of points in Euclidean space with a non-empty interior. We call T an *r-gentile* if T admits an *r-gentiling*, that is, a subdivision of T into $r \geq 2$ sets (*tiles*) $T_1, \ldots, T_r$, such that each of the sets $T_1, \ldots, T_r$ is similar to T. In other words, T is an r-gentile if we can tile it with r smaller copies of itself. This generalizes the concept of *reptiles*, coined by Golomb [6]: a set T is an *r-reptile* if T admits an *r-reptiling*, that is, a subdivision of T into $r \geq 2$ sets $T_1, \ldots, T_r$, such that each of the sets $T_1, \ldots, T_r$ is similar to T *and all sets* $T_1, \ldots, T_r$ *are mutually congruent under translation, rotation and/or reflection.* In other words, T is an r-reptile if we can tile it with r equally large, possibly reflected, smaller copies of itself. Interest in reptile tetrahedra (or triangles, for that matter) exists, among other reasons, because of their application in meshes for scientific computing [1,10]. In this realm techniques such as reptile-based stack-and-stream are well-developed in two dimensions, but three-dimensional space poses great challenges [1].

It is known what triangles are r-reptiles [13] and r-gentiles [4,9] for what r. However, for tetrahedra the situation is much less clear; in fact the identification of reptile and gentile tetrahedra and, even more general, of tetrahedra that tile space, has been a long-standing open problem [12].

The regular tetrahedron does *not* tile space, as its dihedral angles are $\arccos(1/3)$, which is larger than $2\pi/6$ but slightly smaller than $2\pi/5$, so that no number of regular tetrahedra can fill the space around a common edge. Goldberg described all *known* tetrahedra that do tile space [5]. Delgado Friedrichs and Huson characterize all tetrahedra that produce *tile-transitive* tilings [2], but to the best of my knowledge, without the restriction to tile-transitive tilings the problem of identifying all space-filling tetrahedra is still open.

The reptile tetrahedra must be a subset of the tetrahedra that tile space. Matoušek and Safernová argued that r-reptilings with tetrahedra exist if and only if r is a cube number [11]. In particular, it is known that all so-called *Hill tetrahedra* (attributed to Hill [8] by Hertel [7] and Matoušek and Safernová [11]) are 8-reptiles. It has been conjectured that the Hill tetrahedra are the only reptile tetrahedra [7], but this conjecture is false: Sommerville found two non-Hill tetrahedra that tile three-dimensional space [14] and which were recognized as 8-reptiles by Liu and Joe [10]. To the best of my knowledge, the Hill tetrahedra and the two non-Hill tetrahedra from Liu and Joe are the only tetrahedra known to be reptiles, but there might be others. This paper provides a small contribution to the answer to the question: exactly what tetrahedra are reptiles?

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https://doi.org/10.1016/j.disc.2017.10.010
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In mesh construction applications one typically needs to enforce certain quality constraints on the mesh elements. This has motivated studies into *acute tetrahedra* [3]:

Definition 1. A tetrahedron is *acute* if each pair of its facets has a dihedral angle strictly less than $\pi/2$.

All facets of an acute tetrahedron are acute triangles themselves (Eppstein et al. [3], Lemma 2). The Hill tetrahedra, as well as the two non-Hill tetrahedra from Liu and Joe, all have right dihedral angles.[1] Thus, no acute reptile tetrahedra are known.

2. Results

In this note we will prove the following statement, which may serve as evidence that acute reptile tetrahedra are probably hard to find, if they exist at all:

Theorem 1. *Let T be an acute tetrahedron subdivided into $r \geq 2$ acute tetrahedra $T_1, \ldots, T_r$. If the diameter (longest edge) of each tetrahedron T_i is smaller than the diameter (longest edge) of T, then $r \geq 10$.*

In particular we get:

Corollary 1. *No acute tetrahedron is an r-gentile for any $r < 10$.*

With the result from Matoušek and Safernová that r-reptile tetrahedra can only exist when r is a cube number [11], we get:

Corollary 2. *No acute tetrahedron is an r-reptile for any $r < 27$.*

3. The proof

Note that if a tetrahedron T is subdivided into tetrahedra $T_1, \ldots, T_r$ with smaller diameter than T, then at least one tetrahedron T_i, for some $i \in \{1, \ldots, r\}$, must have a vertex v on the longest edge of T. For the proof of Theorem 1 we analyse $\mathcal{S}_v$, the subdivision of an infinitesimal sphere around v that is induced by the facets of T and $T_1, \ldots, T_r$. In such a subdivision, we find:

- faces: each face is either a spherical triangle, corresponding to a tetrahedron T_i of which v is a vertex, or a spherical diangle (also called lune), corresponding to a tetrahedron that has v on the interior of an edge;
- edges: the edges of $\mathcal{S}_v$ are segments of great circles and correspond to facets of $T_1, \ldots, T_r$ that contain v; the angle between two adjacent edges on a face of $\mathcal{S}_v$ corresponds to the dihedral angle of the corresponding facets of a tetrahedron T_i.
- vertices: each vertex of $\mathcal{S}_v$ corresponds to an edge of a tetrahedron T_i that contains v.

Thus, $\mathcal{S}_v$ consists of a spherical diangle D corresponding to T, subdivided into a number of spherical triangles, and possibly some spherical diangles, that correspond to the tetrahedra from $T_1, \ldots, T_r$ that touch v. Below we will see that $\mathcal{S}_v$ must contain at least ten faces (not counting the outer face, that is, the complement of D), which proves Theorem 1.

In what follows, when we talk about diangles and triangles, we will mean *acute*, *spherical* diangles and *acute*, *spherical* triangles on a sphere with radius 1. Note that the faces are diangles or triangles in the geometric sense, but they may have more than two or three vertices on their boundary. More precisely, a diangle or triangle has, respectively, exactly two or three vertices, called *corners*, where its boundary has an acute angle, and possibly a number of other vertices where its boundary has a straight angle. A chain of edges of a diangle or triangle from one corner to the next is called a *side*. Note that $\mathcal{S}_v$ contains at least one triangle, since v is a vertex of at least one tetrahedron T_i. Therefore, in what follows we consider a subdivision $\mathcal{S}$ of a diangle D into a number of diangles and triangles, among which at least one triangle. We call such subdivisions *valid*. Henceforth, we will assume that $\mathcal{S}$ has the smallest number of faces out of all possible valid subdivisions of all possible diangles D. Our goal is now to prove that $\mathcal{S}$ contains at least ten faces.

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[1] One of the non-Hill tetrahedra can be given (modulo similarity transformations) by $A = (-1, 0, 0)$, $B = (0, 1, 0)$, $C = (1, 0, 0)$, $D = (0, 0, \sqrt{1/2})$; the second type of non-Hill tetrahedron is obtained by cutting the first type along the yz-plane. Both types have right dihedral angles along the x-axis. Any Hill tetrahedron can be described as the convex hull of four vertices $A = 0$, $B = v_1$, $C = v_1 + v_2$ and $D = v_1 + v_2 + v_3$, such that the vectors v_1, v_2 and v_3 have the same length and such that the angle between each pair of these vectors is the same, say α [11]. For ease of notation, assume that the tetrahedron is scaled, rotated and reflected such that v_1, v_2 and v_3 have length $\sqrt{2}$, the vertex $C = v_1 + v_2$ lies on the positive x-axis, and the vertex $B = v_1$ lies in the first quadrant of the xy-plane. We use t to denote $\cos\alpha$. Note that we have $t < 1$, otherwise we would have $\alpha = 0$, all vertices would lie on a single line, and they would not be the vertices of a tetrahedron. The condition on the angles of the vectors can now be written as $v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 2t$. Thus we must have $v_1 = (a, b, 0)$ and $v_2 = (a, -b, 0)$ with $a = \sqrt{1+t}$ and $b = \sqrt{1-t} > 0$, so that indeed, $\|v_1\| = \|v_2\| = \sqrt{a^2 + b^2} = \sqrt{2}$ and $v_1 \cdot v_2 = a^2 - b^2 = 2t$. The vector $v_3 = (x, y, z)$ must now satisfy $v_1 \cdot v_3 = v_2 \cdot v_3 \Leftrightarrow ax + by = ax - by$, which, given $b \neq 0$, solves to $y = 0$, and we get $D = v_1 + v_2 + v_3 = (2a + x, 0, z)$. Thus, the face ABC lies in the xy-plane and the face ACD lies in the xz-plane, and these faces meet at a right dihedral angle along the x-axis.

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