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Orthogonally Resolvable Matching Designs

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ABSTRACT

An orthogonally resolvable matching design OMD(n, k) is a partition of the edges of the complete graph K_n into matchings of size k, called blocks, such that the blocks can be resolved in two different ways. Such a design can be represented as a square array whose cells are either empty or contain a matching of size k, where every vertex appears exactly once in each row and column. In this paper we show that an OMD(n, k) exists if and only if $n \equiv 0 \pmod{2k}$ except when $k = 1$ and $n = 4$ or 6.

1. Introduction

We assume that the reader is familiar with the general concepts of graph theory and design theory, and refer them to [3,12]. In particular, the *Lexicographic product* of a graph G with a graph H, denoted $G[H]$, is defined as the graph on vertex set $V(G) \times V(H)$ with $(u_G, u_H)(v_G, v_H) \in E(G[H])$ if $u_G v_G \in E(G)$, or $u_G = v_G$ and $u_H v_H \in E(H)$. In the case where H is the empty graph on w points, we denote the lexicographic product $G[H]$ by $G[w]$. We use K_n to denote the complete graph on n vertices and thus $K_n[w]$ is the complete multipartite graph with n parts of size w. Also, a *matching* on $2k$ vertices is denoted M_k and is defined as a set of k disjoint edges.

Given two graphs G and H, a G-decomposition of H is a partition of the edges of H into graphs isomorphic to G, called *blocks*. The most studied case is when H is the complete graph, in which case we call a G-decomposition of H a *G-design*. A *resolution class* of a G-decomposition of H is a set of blocks which partitions the point set. A G-decomposition of H is called *resolvable* if the set of all blocks can be partitioned into resolution classes. In this case each point appears in the same number of blocks and we call this the replication number and denote it by r. A resolvable G-decomposition of H is also referred to as a *G-factorization of H* and a single class as a *G-factor of H*. We note that each factor is a spanning subgraph of H.

If a G-decomposition of H has two resolutions such that the intersection between any class from one resolution with any class from the other is at most one block, then the decomposition is *orthogonally resolvable*, sometimes called *doubly resolvable*. An orthogonally resolvable G-decomposition of H can be represented by an $r \times r$ array, where each cell is either empty or contains a block of the decomposition. Each row and each column is a resolution class and thus contains each point exactly once.

The simplest case of a G-decomposition of H is when G is a single edge $K_2 \cong M_1$. A resolvable M_1-decomposition is also known as a 1-factorization, which are well studied, see [11]. In particular it is well known that a 1-factorization of K_n exists if and only if n is even and a 1-factorization of $K_{n,n} \cong K_2[n] \cong M_1[n]$ exists for all $n \in \mathbb{Z}^+$.

The study of orthogonal resolutions of designs has a long history. An orthogonal 1-factorization is called a *Room square*, after T. G. Room [10] who studied them in the 1950s. However, the study of Room squares goes back to the original work of

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Kirkman in 1847 [7], where he presents a Room square of order 8. The existence of Room squares was finally settled in 1975 by Mullin and Wallis [9]; for a survey on Room squares see [5].

Orthogonally resolvable K_3-Designs are known as *Kirkman squares*, and have been well studied, see for example [2,4,6,8]. In particular, Mathon and Vanstone [8] showed the non-existence of a Kirkman square of orders $n = 9$ and 15; the existence of Kirkman squares was settled by Colbourn, Lamken, Ling and Mills in [4], with 23 possible exceptions, 11 of which were solved in [6]. Another generalization of G-designs that has been considered is when G is an n-cycle, see [1].

Decompositions of other graphs have also been considered, for example it is easy to see that an orthogonal 1-factorization of $K_2[n]$ is equivalent to a pair of mutually orthogonal Latin squares, which are well known to exist for all $n \neq 2, 6$.

In this paper we consider orthogonally resolvable M_k-decompositions and designs. An orthogonally resolvable M_k-decomposition of G is denoted OMD(G, k), and when $G = K_n$ we write OMD(n, k). We can also define these decompositions in terms of the corresponding square $r \times r$ array.

Definition 1.1. An *OMD* (n, k) is defined as a decomposition of K_n into two orthogonal resolutions with blocks that are isomorphic to M_k.

We can also define such a decomposition in terms of a square $r \times r$ array such that:

1. Each cell is either empty or contains a copy of M_k, where M_k is the matching on k edges.
2. Each row R and each column S contains each element of $V(K_n)$ exactly once.
3. Every pair $x, y \in V(K_n)$ occurs together as an edge of one of the M_k exactly once.

A consequence of conditions 1, 2 and 3 is that $r = n - 1$, as well as the following necessary condition for the existence of any OMD(n, k), which comes from counting the edges and vertices of K_n and M_k.

Lemma 1.2. *An OMD*(n, k) *exists only if* $n \equiv 0 \pmod{2k}$.

Room squares are thus OMD$(n, 1)$ and we state the result of Mullin and Wallis in the following theorem using the language of OMDs.

Theorem 1.3 (*[9]*)**.** *An OMD*$(n, 1)$ *exists if and only if* n *is even and* $n \neq 4, 6$.

In the next section we consider some small cases and then we provide a recursive construction and prove our main theorem, which we state here.

Theorem 1.4. *There exists an OMD*(n, k) *if and only if* $n \equiv 0 \pmod{2k}$ *except when* $k = 1$ *and* $n = 4$ *or* 6.

2. Small cases

In this section we consider some of the ingredients we will need as well as OMD(mk, k) for small values of m. We generally work on the corresponding $r \times r$ array, where the rows and columns are indexed by $\mathbb{Z}_r$. We begin by defining some terms that we will find useful.

Definition 2.1. A *transversal* of an *OMD*(n, k) is a set of r cells, which contains exactly one cell from each row and one cell from each column, such that each point appears exactly once in one of the cells.

We note that in general a transversal will contain empty cells. For example, in an OMD$(2n, 1)$ a transversal will have n non-empty cells.

Definition 2.2. We say that an *OMD*(n, k) has a *hole* of size m if it contains a square subarray of side m which is empty.

Lemma 2.3. *There exists an OMD*$(M_1[k], k)$ *for all* $k \in \mathbb{Z}^+$.

Proof. Note that $M_1[k] \cong K_{k,k}$ and each 1-factor of $K_{k,k}$ is isomorphic to M_k. We obtain the design by placing the 1-factors of a 1-factorization of $K_{k,k}$ down the diagonal of the square. □

Lemma 2.4. *There exists an OMD*$(2k, k)$ *with a transversal and a hole of size* $k - 1$ *for all* $k \in \mathbb{Z}^+$.

Proof. We note that each 1-factor of K_{2k} is isomorphic to M_k. We obtain the design by placing the 1-factors of a 1-factorization of K_{2k} down the diagonal of the square. The back diagonal $(2k - 2 - i, i)$, $0 \leq i < 2k - 1$, is a transversal, with $i = k - 1$ being the only non-empty cell. Further, the upper right $(k - 1) \times (k - 1)$ subarray is empty. □

Lemma 2.5. *There exists an OMD*$(4k, k)$ *for all* $k > 1$.