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On the unimodality of independence polynomials of very well-covered graphs

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ABSTRACT

The independence polynomial $i(G, x)$ of a graph G is the generating function of the numbers of independent sets of each size. A graph of order n is very well-covered if every maximal independent set has size $n/2$. Levit and Mandrescu conjectured that the independence polynomial of every very well-covered graph is unimodal (that is, the sequence of coefficients is nondecreasing, then nonincreasing). In this article we show that every graph is embeddable as an induced subgraph of a very well-covered graph whose independence polynomial is unimodal, by considering the location of the roots of such polynomials.

1. Introduction

A subset S of the vertex set of a (finite, undirected) graph G is said to be *independent* if S induces a graph with no edges. The *independence polynomial* of a graph G is defined to be

$$i(G, x) = \sum_{k=0}^{\alpha} i_k x^k,$$

where i_k is the number of independent sets of size k in G and $\alpha = \alpha(G)$, the *independence number* of G, is the size of the largest independent set in G. The independence polynomial is the generating function of the *independence sequence* $\langle i_0, i_1, \ldots, i_\alpha \rangle$. The independence polynomial of a graph has been of considerable interest [2,3,14–16,18,21–23] since it was first defined by Gutman and Harary in 1983 as a generalization of the matching polynomial.

For many graph polynomials (such as matching [17], chromatic [19,27] and reliability [11,20] polynomials), the (absolute value of the) coefficient sequence, under a variety of bases expansions, have long been conjectured to be (or proven to be) *unimodal*, that is, nondecreasing then nonincreasing. We say that a polynomial is unimodal if its sequence of coefficients is unimodal.

What can we say about the unimodality of independence polynomials? They certainly form a sequence of positive integers. Alavi et al. [1] showed, in general, that the independence sequence $\langle i_k \rangle$ of a graph G can be far from unimodal, for example, the graph $K_{25} + 4K_2$ has independence sequence $\langle 1, 33, 24, 32, 16 \rangle$. More examples of graphs with nonunimodal independence sequences can be found in [1].

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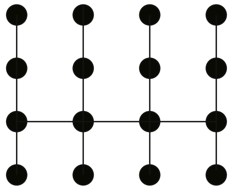

Fig. 2.1. The double leafy extension of P_4.

However, there are classes of graphs for which the independence coefficients are indeed unimodal. In a beautiful paper [10], Chudnovsky and Seymour proved that the coefficients of the independence polynomials of *claw–free* graphs (that is, those without an induced star on 4 vertices) are unimodal.

Another highly structured family of graphs with respect to independence is *well-covered* graphs, those whose maximal independent sets all have the same size (complete graphs and the 5-cycle are examples). The structure of such graphs has attracted considerable attention in the literature, with characterizations for those of high girth [13]. In [6], the authors conjectured that the independence coefficients of well-covered graphs were unimodal, and showed that every graph G can be embedded as an induced subgraph of such a well-covered graph. However, Michael and Traves [26] later disproved the conjecture. A conjecture due to Alavi et al. [1] that is still open is that the independence polynomial of a tree is unimodal.

Finally, Levit and Mandrescu [24] amended the original unimodality conjecture on well-covered graphs as follows. A *very well-covered* graph G of order n (that is, on n vertices) is a well-covered graph for which every maximal independent set has size $n/2$; for example, the complete bipartite graphs $K_{m,m}$ are very well-covered. Other examples are afforded by the following construction. Let G be any graph. Form G^*, the *leafy extension* of G (sometimes also called the *corona* of G with K_1) from G by attaching, for each vertex v of G a new vertex v^* to v with an edge (such a vertex is called a *pendant* vertex); leafy extensions are always very well-covered (more about that shortly).

Levit and Mandrescu conjectured that the coefficients of the independence polynomials of a very well-covered graph are unimodal, and to date, the conjecture remains open. Some partial results have been proven on the tail of independence sequences of very well-covered graphs [25] and the first $\lceil \frac{\alpha}{2} \rceil$ terms have been shown to be nondecreasing for well-covered graphs [26]. The conjecture is known to hold when $\alpha(G) \leq 9$ [25] and for leafy extensions of any graph G where $\alpha(G) \leq 8$ [9], or where G is a path or star [22]. In this paper we shall show that Levit and Mandrescu's conjecture holds for some iterated leafy extensions of *any* graph G.

2. Unimodality of independence polynomials of leafy extensions and sectors in the complex plane

The leafy extension G^* of any graph $G = (V, E)$ of order n is always very-well-covered. Clearly, $\alpha(G^*) \leq n$, as the graph has a perfect matching (and no independent set can contain two vertices that are matched). Moreover, $\alpha(G^*) = n$ as any independent set I of G can be extended to one in G^* by adding in any subset of $(V - I)^* = \{v^* : v \in V - I\}$. It follows (see also [22]) that if $i(G, x) = \sum i_k x^k$, then

$$\begin{aligned} i(G^*, x) &= \sum i_k x^k (1+x)^{n-k} \\ &= (1+x)^n \cdot i\left(G, \frac{x}{1+x}\right). \end{aligned} \tag{1}$$

For a graph G and positive integer k, let G^{k*} denote the *kth iterated leafy extension* of G, that is, the graph formed by recursively attaching pendant vertices, k times:

$$G^{k*} = \begin{cases} G^* & \text{if } k = 1, \\ (G^{(k-1)*})^* & \text{if } k \geq 2. \end{cases}$$

Fig. 2.1 shows the graph P_4^{2*}.

We can extend formula (1) to higher iterations of the $*$ operation as follows.

Proposition 2.1. *For any graph G of order n and any positive integer k,*

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}.$$

Proof. We proceed by induction on k, the number of iterations of the $*$ operation. The base case follows directly from (1), so we can assume that the result holds for some $k \geq 1$, i.e.,

$$i(G^{k*}, x) = i(G, \tfrac{x}{kx+1})(kx+1)^n \prod_{\ell=1}^{k-1} (\ell x + 1)^{n2^{k-\ell-1}}.$$

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