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Enumeration formulas for standard Young tableaux of nearly hollow rectangular shapes

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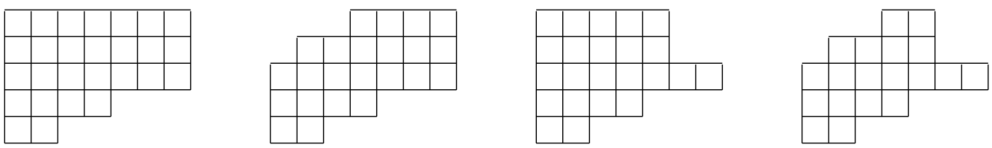
ABSTRACT

This paper considers the enumeration problem of a generalization of standard Young tableau (SYT) of truncated shape. Let $\lambda \setminus \mu|\{(i_0, j_0)\}$ be the SYT of shape λ truncated by μ whose upper left cell is (i_0, j_0), where λ and μ are partitions of integers. The summation representation of the number of SYT of the truncated shape $(n+k+2, (n+2)^{m+1}) \setminus (n^m)|\{(2, 2)\}$ is derived. Consequently, three closed formulas for SYT of hollow shapes are obtained, including the cases of (i). $m = n = 1$, (ii). $k = 0$, and (iii). $k = 1, m = n$. Finally, an open problem is posed.

1. Introduction

The enumeration of standard Young tableau (SYT) of truncated shapes was recently considered in enumerative combinatorics. SYT of truncated shape was introduced in [4], and a new interpretation was given in [2]. The works in the literature discussed counting the number of SYTs with some cells being deleted from the northeastern corner. Adin et al. have obtained formulas for rectangular and shifted staircase shapes truncated by a square or nearly a square [1]. G. Panova derived the formula for rectangle truncated by a staircase [5]. P. Sun obtained formulas for skew SYTs truncated by a rectangle [8], and proved the product formula conjectured in [1] for a square truncated by two cells [9]. Problems and advances in the enumeration of SYT of truncated shape are discussed in Adin and Roichman's recent survey paper [3].

Recall that a partition λ of a positive integer $|\lambda|$ is a non-increasing sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $|\lambda| = \lambda_1 + \cdots + \lambda_d$. A Young diagram $[\lambda]$ of shape λ is a left-justified array of $|\lambda|$ cells, with row i (from top to bottom) containing λ_i cells, namely $[\lambda] := \{(i, j) | 1 \le i \le d, 1 \le j \le \lambda_i\}$. An SYT of shape λ is a labeling by $\{1, 2, \ldots, |\lambda|\}$ of the cells in the Young diagram such that each row and column is increasing (from left to right, and from top to bottom, respectively). The number of SYT is given by the well-known hook-length formula. Let λ, μ, ν are partitions of positive integers, a skew shape λ/μ is the collection of cells which belong to λ but not μ when drawn with coinciding upper left corners, and a diagram of truncated shape $\lambda \setminus \nu$ is a left-justified array of cells where ν_i cells are deleted from the end of row i. An SYT of truncated shape $\lambda \setminus \nu$ is a filling of the corresponding truncated diagram with the integers from 1 to $|\lambda| - |\nu|$ such that each row and column is increasing [3,7]. For examples, $\lambda = (7^3, 4, 2)$, $\mu = (3, 1)$, and $\nu = (2^2)$, these types are illustrated as



λ λ/μ $\lambda\setminus\nu$ $\lambda/\mu\setminus\nu$

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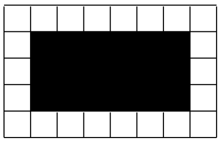
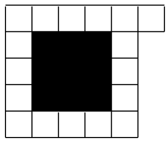
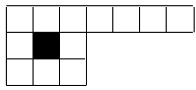
We consider a new kind of SYT of truncated shape $\lambda \setminus \mu|\{(i_0, j_0)\}$—the cells of shape μ are deleted from the diagram of shape λ, with the upper left cell of μ being in row i_0 and column j_0 of λ.

Definition 1. Let $j_0 \geq 1$, if $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_r)$ are partitions such that $j_0 + \mu_j - 1 \leq \lambda_j$ for every j, then the diagram of truncated shape $\lambda \setminus \mu|\{(i_0, j_0)\}$ is the set

$$D = \{(i, j) \in [\lambda], (i_0 + s, j_0 + t) \notin [\lambda], 0 \leq s \leq r - 1, 0 \leq t \leq \mu_{s+1} - 1\}. \tag{1}$$

It is clear that $\lambda \setminus \mu|\{(1, 1)\}$ is the ordinary skew shape λ/μ, and the truncated shape $\lambda \setminus \nu = (7^3, 4, 2) \setminus (2^2)$ in above is written to be $(7^3, 4, 2) \setminus (2^2)|\{(1, 6)\}$. Generally the diagram of truncated shape $\lambda \setminus \mu|\{(i, j)\}$ is non-line-convex, and the enumeration of SYT of non-line-convex shape is a very recent subject of study [3]. In this paper we discuss the enumeration problem of SYT of nearly hollow rectangular shapes, the closed formulas are derived for the SYT of the following shapes:

$(n + 2)^{m+2} \setminus (n^m)|\{(2, 2)\}$ $\qquad$ $(n + 3, (n + 2)^{n+1}) \setminus (n^n)|\{(2, 2)\}$ $\qquad$ $(k + 3, 3^2) \setminus (1)|\{(2, 2)\}$

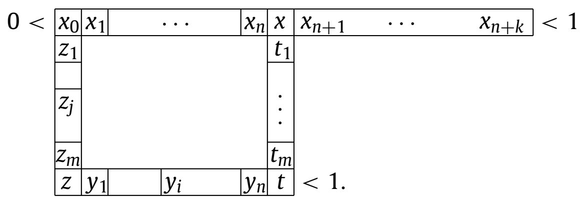
In Section 2 we derive the summation representation of the number of SYT of truncated shape $(n + k + 2, (n + 2)^{m+1}) \setminus (n^m)|\{(2, 2)\}$ $(k \geq 0, m, n \geq 1)$, and our method is multiple integration based on the order statistics model of SYT in [8]. In Section 3 we prove the main results: (i). If $m = n = 1$, the number of SYT of the simplest hollow shape $(k+3, 3^2)\setminus(1)|\{(2, 2)\}$ is $\frac{k+5}{10}\binom{k+2}{2}\binom{k+9}{2}$. (ii). If $k = 0$, the number of SYT of hollow rectangle $(n+2)^{m+2} \setminus (n^m)|\{(2, 2)\}$ is $\binom{2n+2m+2}{n+m+1} - \binom{2n+2m+2}{n-1} - \binom{2n+2m+2}{m-1}$. (iii). If $k = 1$ and $m = n$, the number of SYT of nearly hollow square $(n + 3, (n + 2)^{n+1}) \setminus (n^n)|\{(2, 2)\}$ is $\frac{4n+5}{2}\left[\binom{4n+2}{2n+1} - 2\binom{4n+2}{n-1}\right]$. In Section 4 we give a conjecture, that the number of shifted SYT of hollow staircase shape should be the convolution of Catalan triangle.

2. A combinatorial summation

The order statistics model in [8] implies that the enumeration formula for SYT (even of non-line-convex shape) can be derived by using multiple integration and combinatorial summation. That is, for $k \geq 0$, $m, n \geq 1$, the number $h_{n+2,k,m+2}$ of SYT of truncated shape $(n + k + 2, (n + 2)^{m+1}) \setminus (n^m)|\{(2, 2)\}$ is

$$h_{n+2,k,m+2} = (2n + 2m + k + 4)! \cdot I_{n,k,m} = (2n + 2m + k + 4)! \int \cdots \int_{D_{n,k,m}} dx_r y_r z_r t_r, \tag{2}$$

where $D_{n,k,m}$ is the following SYT-type integral domain (the variables are increasing from left to right and from top to bottom):



Considering $y_{i-1} < x < y_i$ and $z_{j-1} < x < z_j$, $(1 \leq i \leq n + 1, 1 \leq j \leq m + 1, y_0 = z_{m+1} = z)$, $D_{n,k,m}$ is decomposed into $D_1(i)$ and $D_2(j)$ that are illustrated as follows:

$$\begin{array}{ccccccccc}
0 < x_0 & < & x_1 < \cdots < x_{i-1} < \cdots < x_n & < x < x_{n+1} < \cdots < x_{n+k} < 1 \\
\wedge & & \wedge \quad \wedge \quad \wedge & \wedge \\
z_1 < \cdots < z_m < z & < & y_1 < \cdots < y_{i-1} & < x < y_i < \cdots < y_n \\
& & & \wedge \qquad\quad \wedge \\
& & & t_1 < \cdots < t_m < t < 1
\end{array}$$

$$D_1(i), 1 \leq i \leq n + 1,$$

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